The Role of Concept Images in Developing Statistical Understanding

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Troublesome concepts

Interpreting

- p-values
- confidence intervals
- Bonferroni correction
- z-cores
- confounding
- rate of change

Drawing conclusions

- Generalizations
- Accepting, rejecting the null

Interactions

- variability and sample size
- Association and co-variation

Confusing

- Population vs sample
- Binomial distribution vs sampling distribution
- Correlation coefficient and coefficient of determination
- sample distribution and population distribution
- Central Limit Theorem vs normal calculations

Other

- Adding variables in regression without regard to magnitude/unit
- mean of a frequency distribution
- Careless with and/or

The dilemma: students

- often miss the big picture and see statistics as a series of disconnected topics.
  - can calculate a standard deviation and a standard error but do not understand how these concepts are related (both in terms of similarities and differences) and often confuse the concepts
- fail to make the connections between fundamental concepts, such as sample, population, sampling distributions, and sampling variability.
- tend to memorize, for example, the steps in hypothesis testing without understanding the process.
- Eventually become overwhelmed… notation, language, similarities/differences that are not sorted out … (Doorn & Obrien, 2007)

Myths about understanding basic concepts in statistics (Rumsey, 2002)

- Calculations demonstrate understanding of statistical ideas.
- Formulas help students understand the statistical idea.
  - "We too often ignore broad ideas in our rush to convey technical content. We spend too much time calculating and too little time discussing. In short, we are too narrow." (Moore, 1998)
- Students who can explain things in statistical language demonstrate their understanding of a statistical idea.
  - present ideas using relevant and usable language connecting the big ideas with common threads (Rumsey, 2002)

Session Goals

- Consider research supporting visual images to build conceptual knowledge and identify ways in which this might occur,
- Understand slowing the rush to formulas/procedures can enable students to build understanding that will transfer as ideas become more complex,
- Recognize strategic use of interactive dynamic technology can create a learning environment for fostering the development of conceptual understanding,
- Identify characteristics of activities supporting the development of robust images of statistical concepts,
- Identify misunderstandings of statistical ideas and examine how building concept images with interactive dynamic technology can help students avoid these.

What image do you think your students have in their heads when they think about the mean of a set of data?

What image do you think your students have in their heads when they think about the mean of a set of data?

Variability?

A distribution?

Random sample?

A concept image

- can be described as the total cognitive structure including the mental pictures and processes associated with a concept built up in students' minds through different experiences associated with the ideas (Tall & Vinner 1981).
- is necessary to fluently and effectively reason with and apply ideas; without a coherent mental structure, students are left to construct an understanding based on ill formed and often misguided connections and images (Oehrtman 2008).
- intuition inherent in concept images dominates the conceptual learning (Rösken & Rolka, 2007)

Definitions and concept images

- A concept image is not usually built on definitions but essentially determined by typical examples (Vinner & Dreyfus, 1989)
- The concept definition does not seem to play any role when students are working on problems (Vinner, 1994)
- Explanations for concepts will easily be forgotten if students are not able to develop own ideas and associations.
- Learning a new concept requires forming a comprehensive concept image but important aspects of a mathematical concept may not be adequately represented (Rösken & Rolka, 2007)

Definitions and concept images

Concept development refers to the process of unfolding, exploring, and understanding concepts as students engage in experiences designed to move them towards an accurate definition of a concept that is accepted by the field

Visualization

- is a means to develop understanding (Presmeg, 1994).
- includes processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics (Presmeg, 1997).
- provides students with "live" visualizations of a concept that have the potential to enable students to build robust images of the properties, processes and relationships connected to the concept (Drijvers, 2015).
- allows students to link multiple representations of problem scenarios – visual, symbolic, numeric and verbal – and to connect these representations to support understanding (Sacristan et al., 2010).

Interactive dynamic technology

- has the potential to help students build robust concept images in mathematics. Experiences of taking purposeful actions with immediately visual consequences can create particularly compelling dynamic images that may be more powerful than static images (based on Zull, 2002; Michael & Modell, 2003).
- Students can evoke "movie clips" of dynamic experiences that replay in their minds when they encounter words, graphs, equations,… related to a concept such as variable, expression, equation, solution
- From "I remember seeing this…" To "I remember seeing this *happening*…")

Action Consequence Principle

- The learner deliberately takes a statistical (mathematical) action, observes the consequences, and reflects on the statistical implications of the consequences (Dick & Burrill, 2009).

- Interactive dynamic technology provides an opportunity for students to visualize the action and the consequences, which can enable them to create a dynamic mental image of the concept.

Instructional activities supporting the development of concept images

- The underlying structure that is the target for student learning should be reflected in the actions they do.
- Students' actions should be repeated and organized with provisions for feedback and ways to respond to this feedback.
- Students should repeat these actions in structurally similar problems in a variety of contexts to develop a robust abstraction of the concept. (Oehrtman, 2008)
- Emphasize statistical literacy, quantitative reasoning and making sense of ideas and results (GAISEII)

Outline

- Background
- Explore the development of one concept
- Examples of interactive applets supporting the development of a concept image
- Consider how to think about creating a robust concept image
- Inference
- Resources/suggestions

A physical model

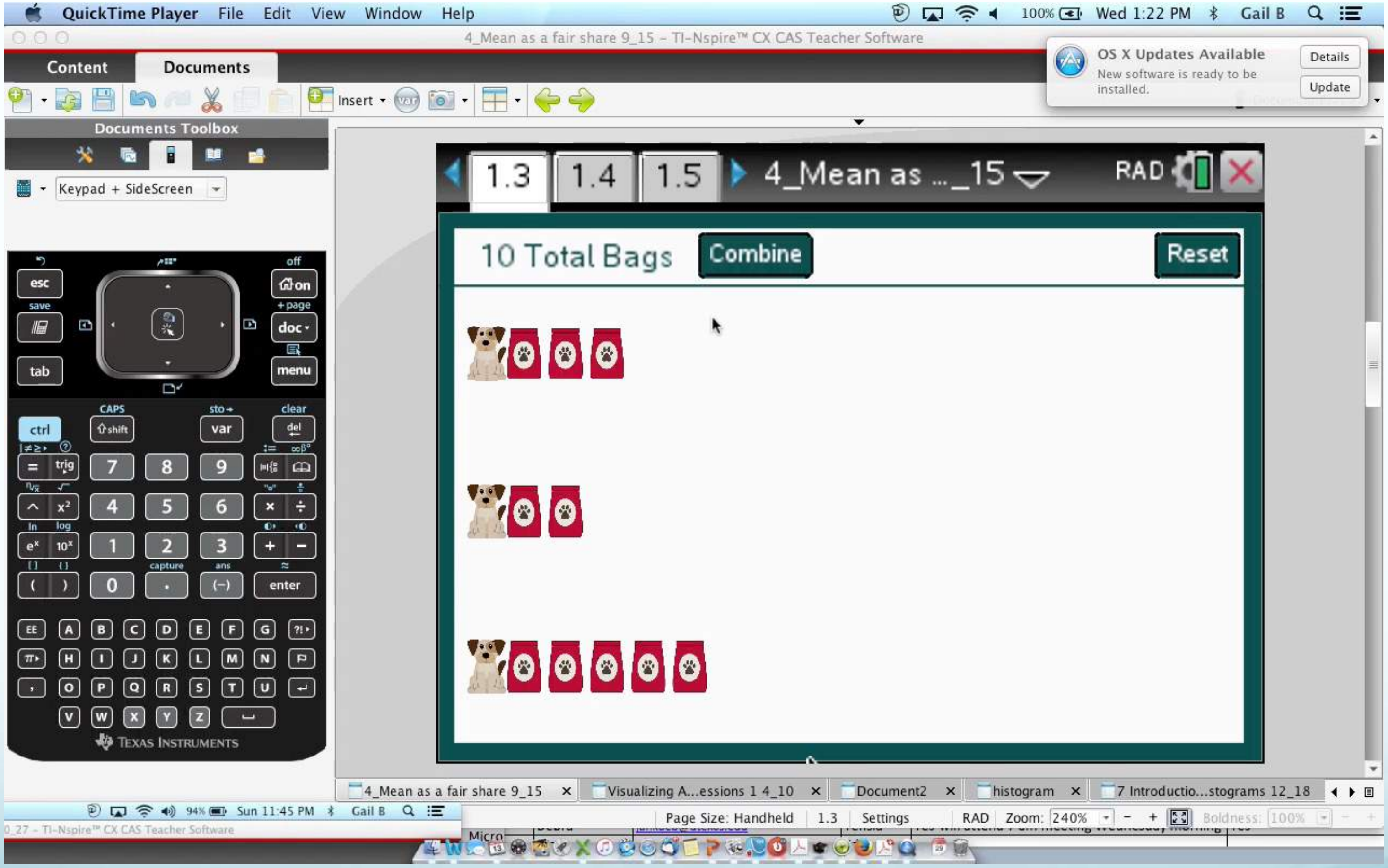
- Hand each of student in a group of four or five (as a class demo or in groups of four or five) a random number of four different colored sticky notes. The task: divide them up in a “fair” way.

Mean as fair share

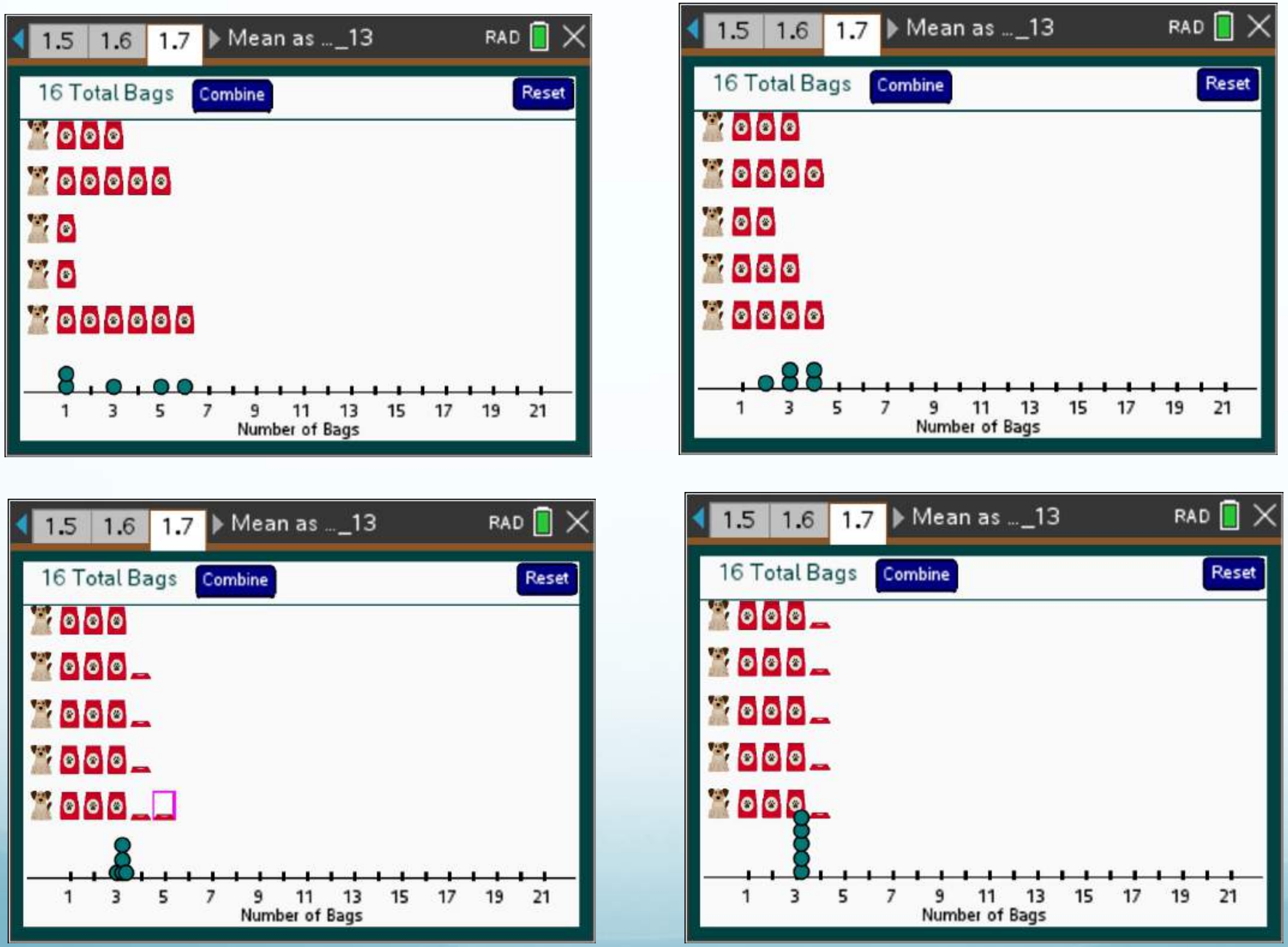
" Leveling"

Pooling and dividing the pool

Mean as Fair Share



A subtle shift: graphing fair share on a number line



Building Concepts: Statistics and Probability, 2016

A Statistical Exploration

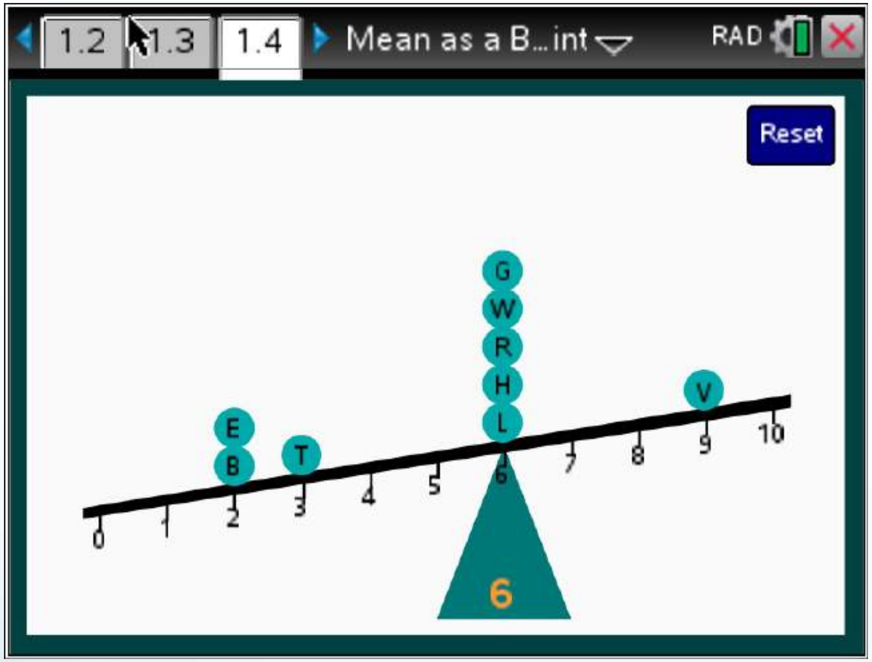
The total number of goals scored by all the teams in a tournament is 54. If the teams were fairly matched, they would have scored 6 goals each.

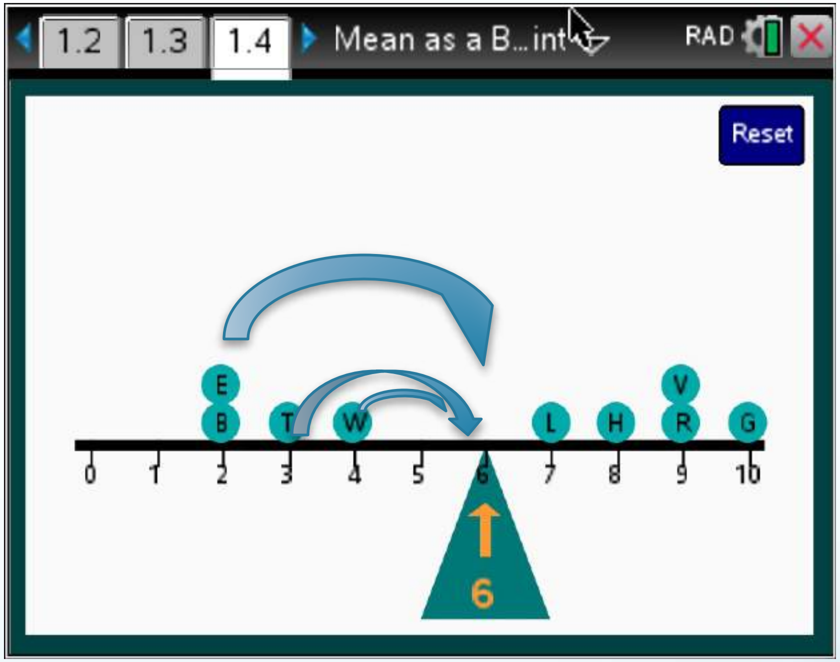
In the actual tournament one team scored 10 goals, another scored 2, another 4 and no team scored 6 goals. Make a distribution of the possible scores of the teams – given that you know the total number of goals scored by all the teams is 54, and every team scored at least one goal, with no team having more than 10 goals.

Adapted from Kader & Memer, 2008

Which tournament has the "least evenly matched teams"? The most? Rank the teams in terms of least to most evenly matched.

Mean as balance point





Misconception: view variability locally not from measure of center

Building Concepts: Statistics and Probability, Mean as Balance Point, 2015

Students should have experiences that

- Create a mental construct or image of the concept as basis for thinking
- Add to this image in robust ways by revisiting concepts in different contexts and connecting ideas
- Build understanding and confront misconceptions
- Include attention to metacognition and flexibility in thinking

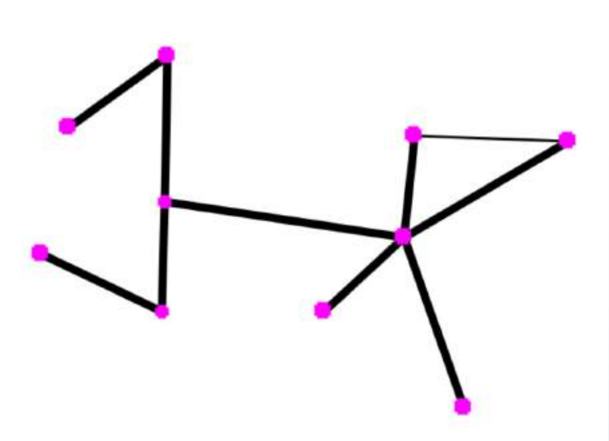
Building a concept image of mean

Summary measures for a distribution

Shift to summary plot

Mean as
fair share

Mean as
pooling



Mean as
balance point

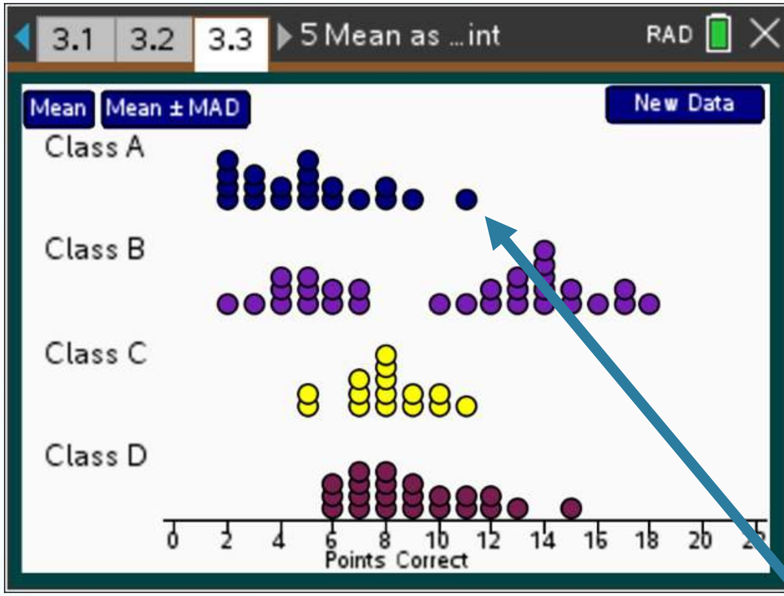
Deviations from mean

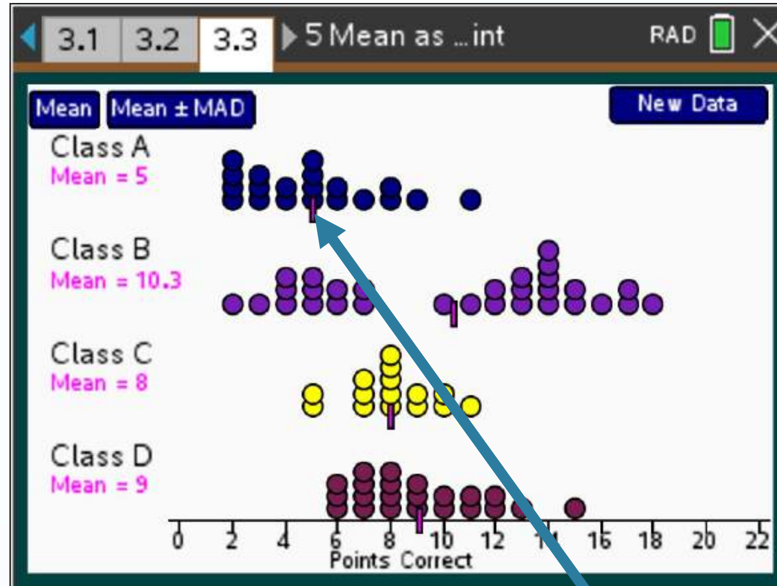
Mean absolute deviation

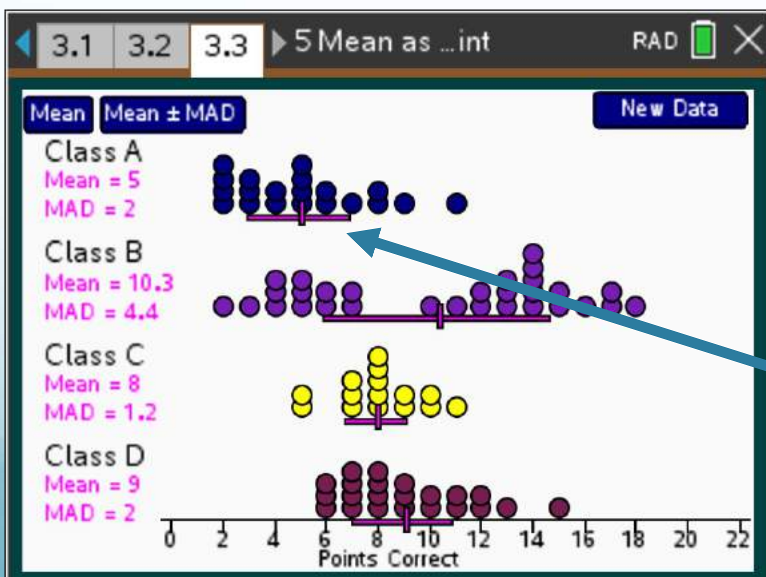
Standard deviation

Outliers

Exploring means & deviations







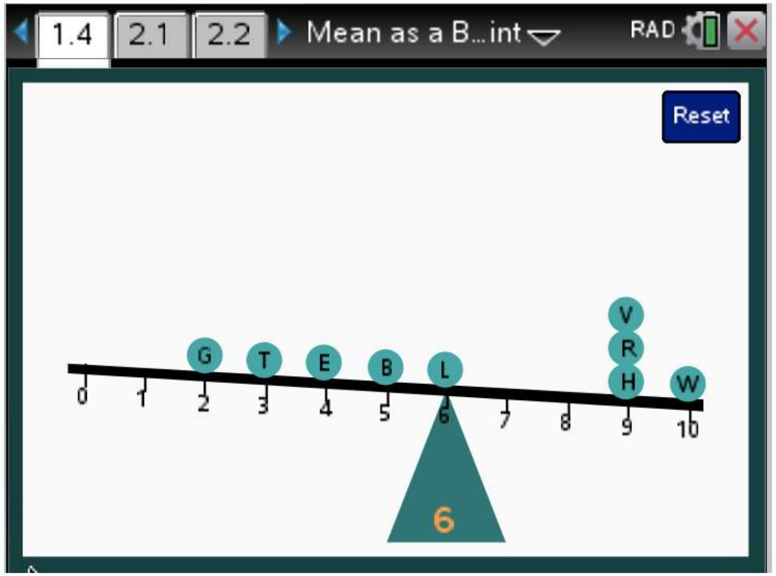
The distribution

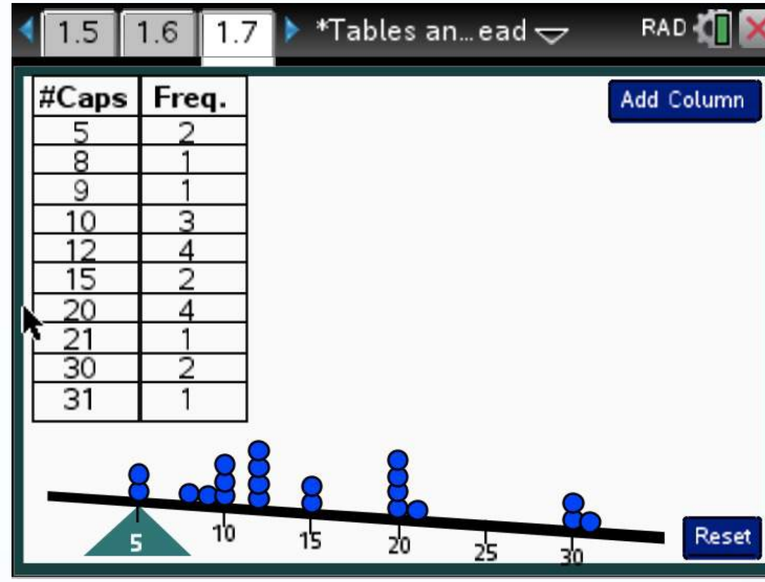
Mean as measure of center

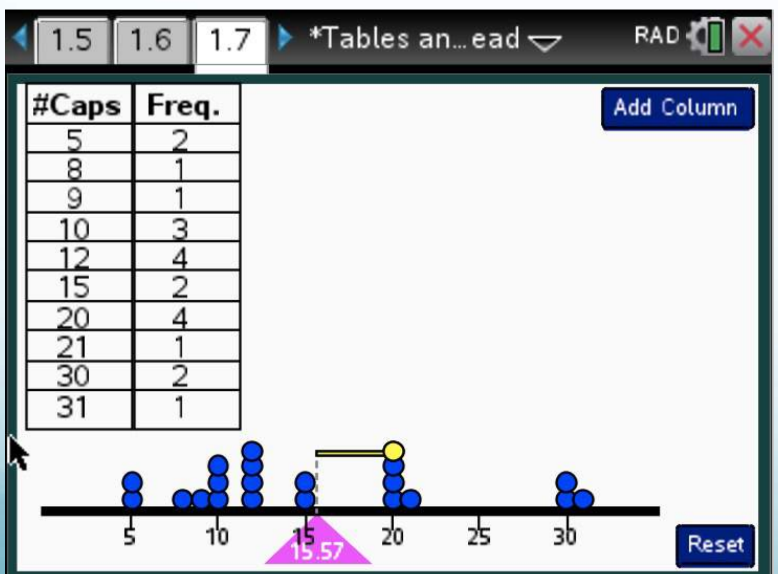
Mean absolute deviation as a measure of variability

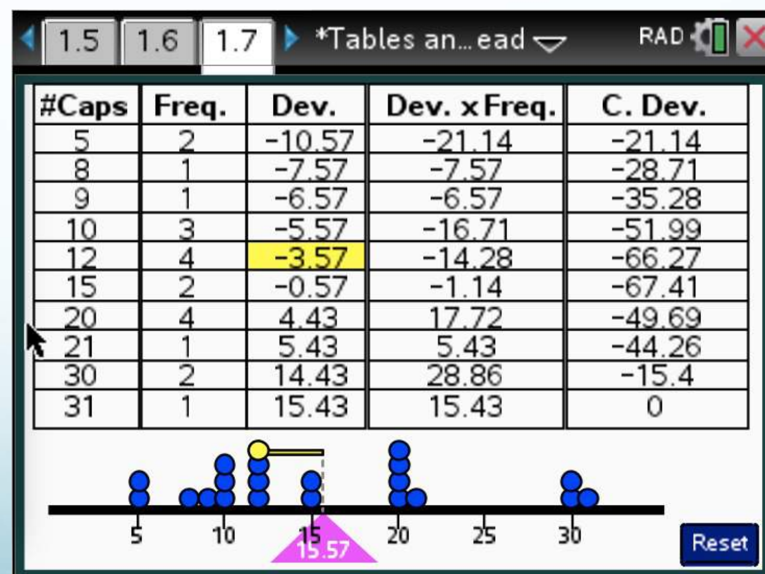
Building Concepts: Statistics and Probability, Mean as Balance Point, 2015

Revisiting deviations - typical distance from mean









Building Concepts: Statistics and Probability, Mean as Balance Point; Tables and measures of center and spread, 2015

A progression

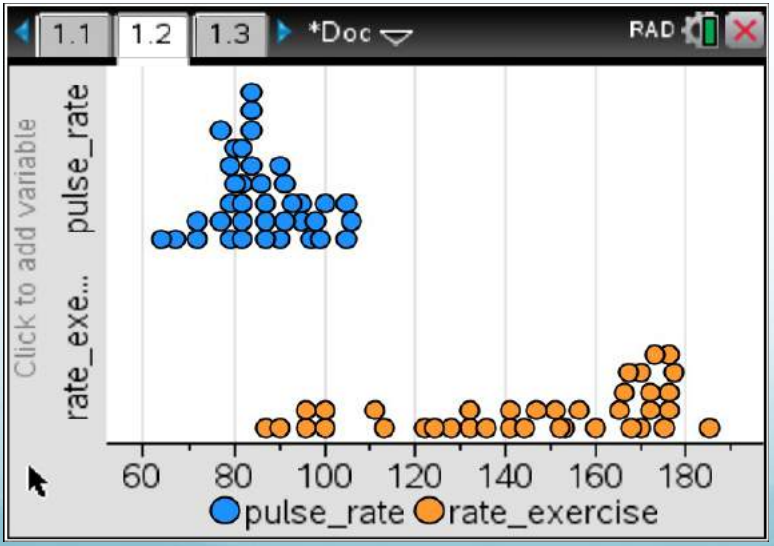
- Transforming the concept of MAD to standard deviation

How would you find the standard deviation in the number of goals for teams with scores for the tournament of 2, 6, 8, 2, 8, 7, 3, 10?

Revisiting mean and MAD

The figure shows the distribution of children's pulse rates both before and after exercise.

a) Estimate the mean for each distribution and explain what this value tells you.
b) Write at least three sentences comparing the pulse rates of the children before and after exercise.



Revisiting mean and standard deviation/MAD

About 80% of the population has brown eyes.

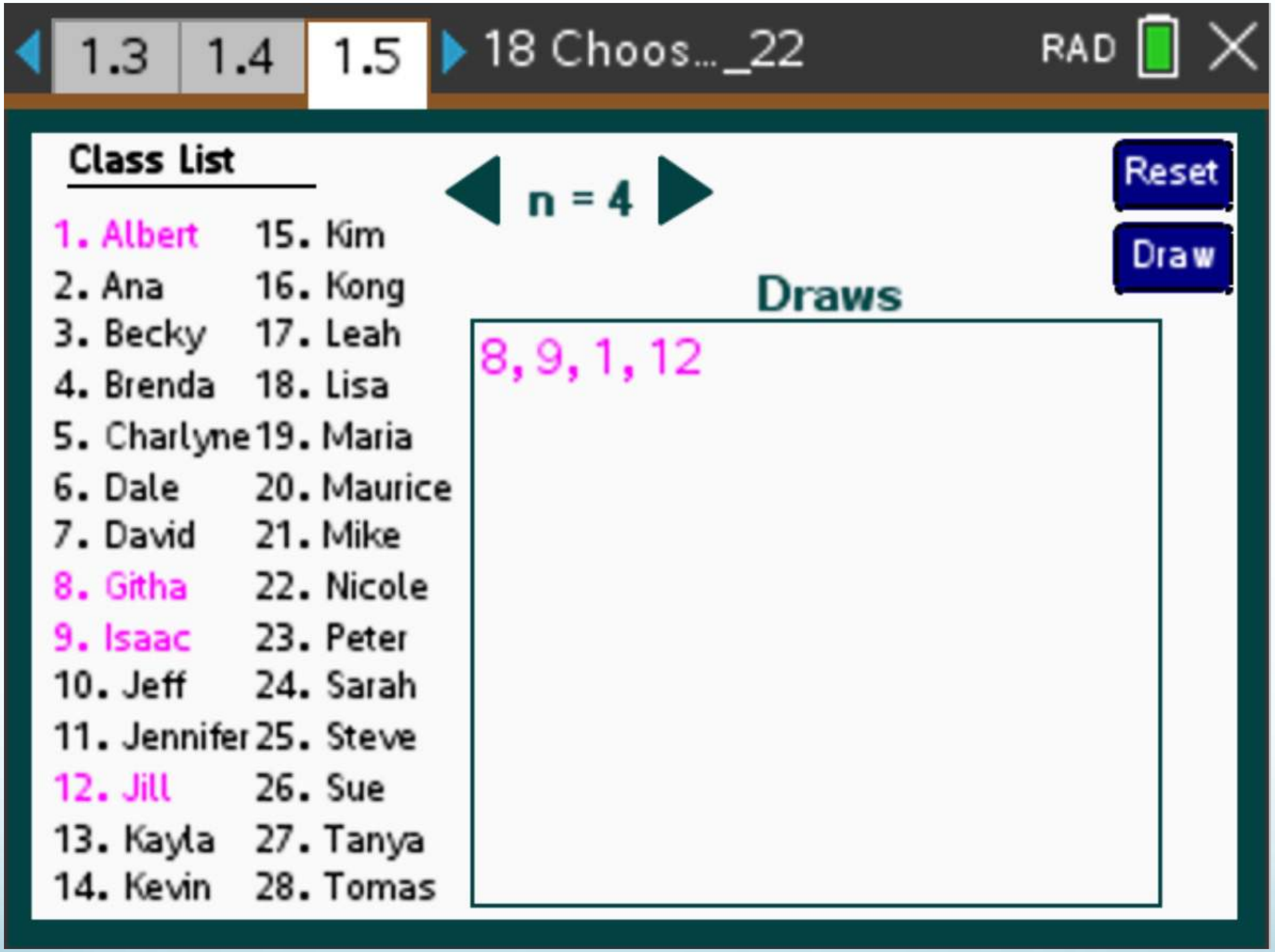
a) Describe the sampling distribution for the number of people with brown eyes you might find in a random sample of 20 people. (Remember to think about shape (including location), center and variability)

b) Describe how the sampling distribution would change if the number of people in the random sample was 50.

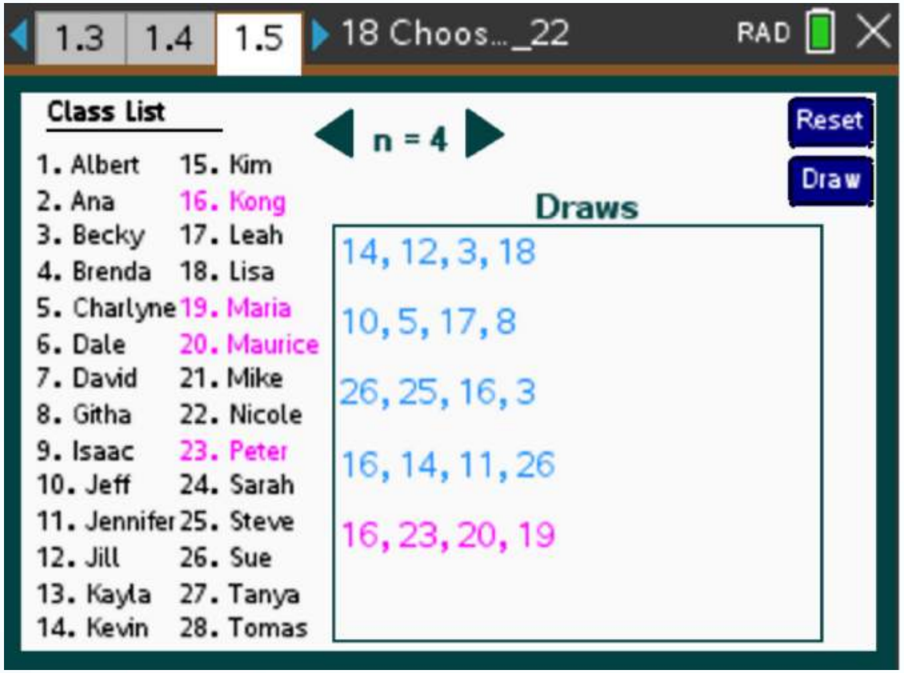
Concept images

- Mean
- Randomness and chance
- Distributions

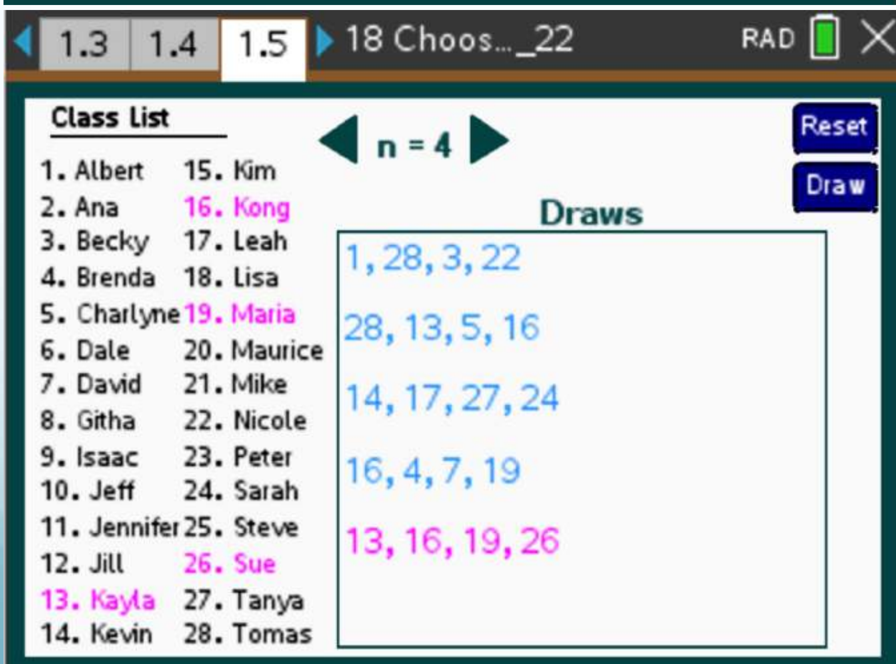
What is a random sample?

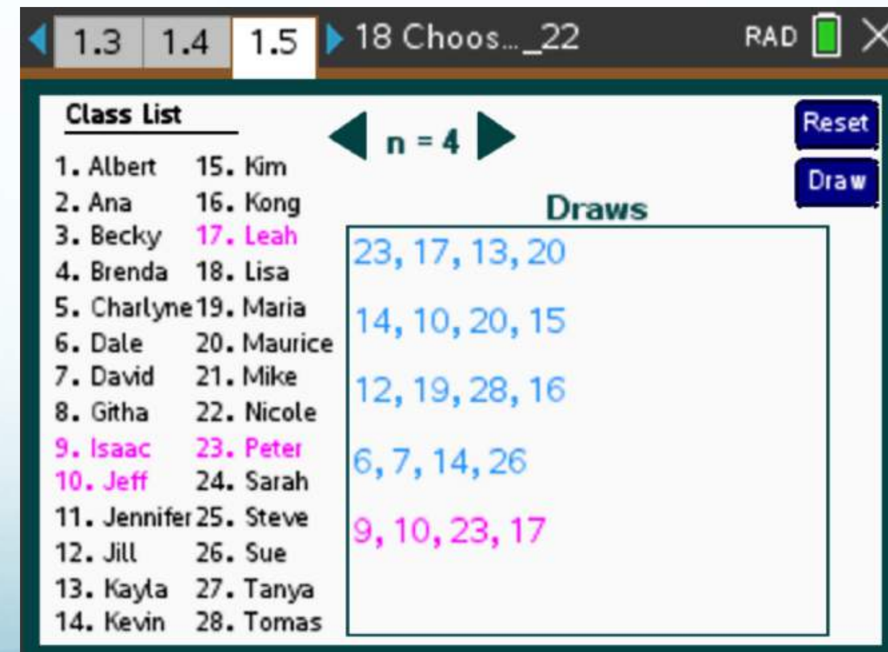


Building Concepts: Statistics and Probability, Choosing Random Samples, 2016



What do you notice?
Is random "fair" in the short term?

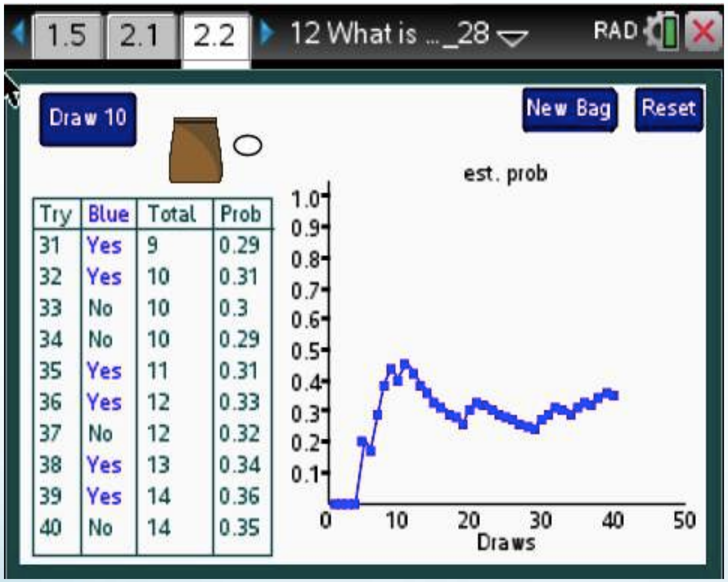


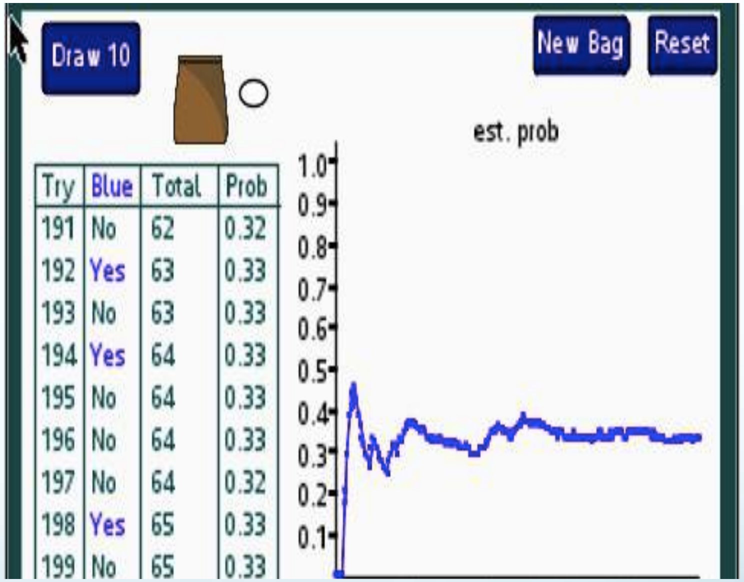


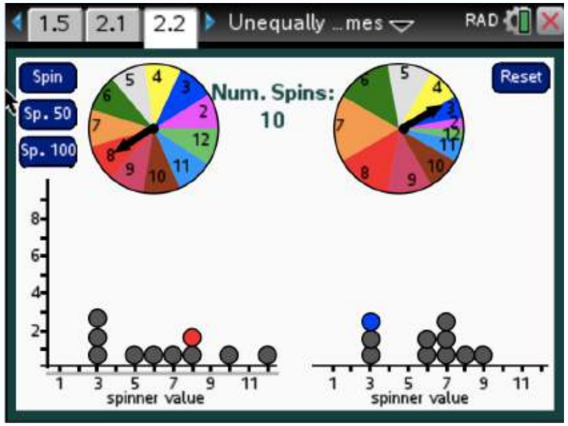
Building Concepts: Statistics and Probability, Choosing Random Samples, 2016

Long run relative frequency

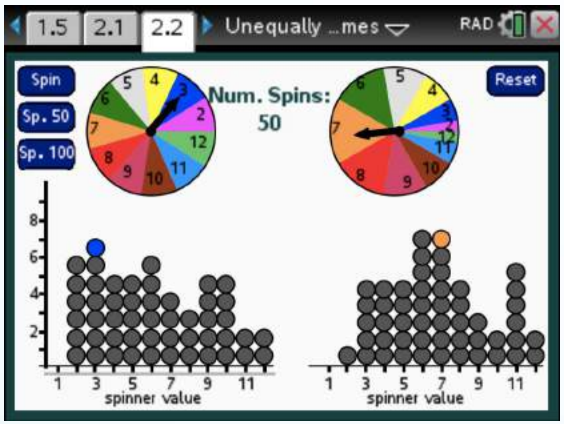
What proportion of the chips in the bag are blue?



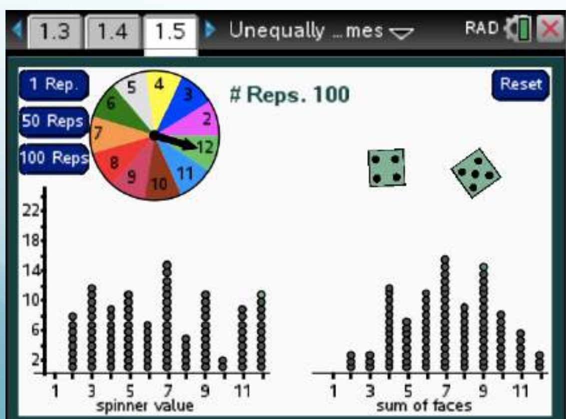


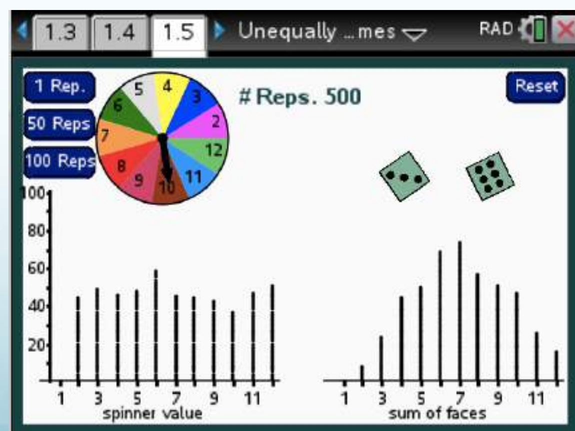


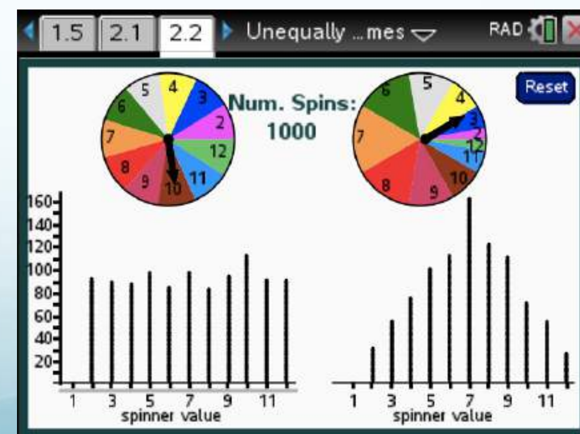
Revisiting randomness



The distribution of outcomes "settle down" as the number of trials increases.







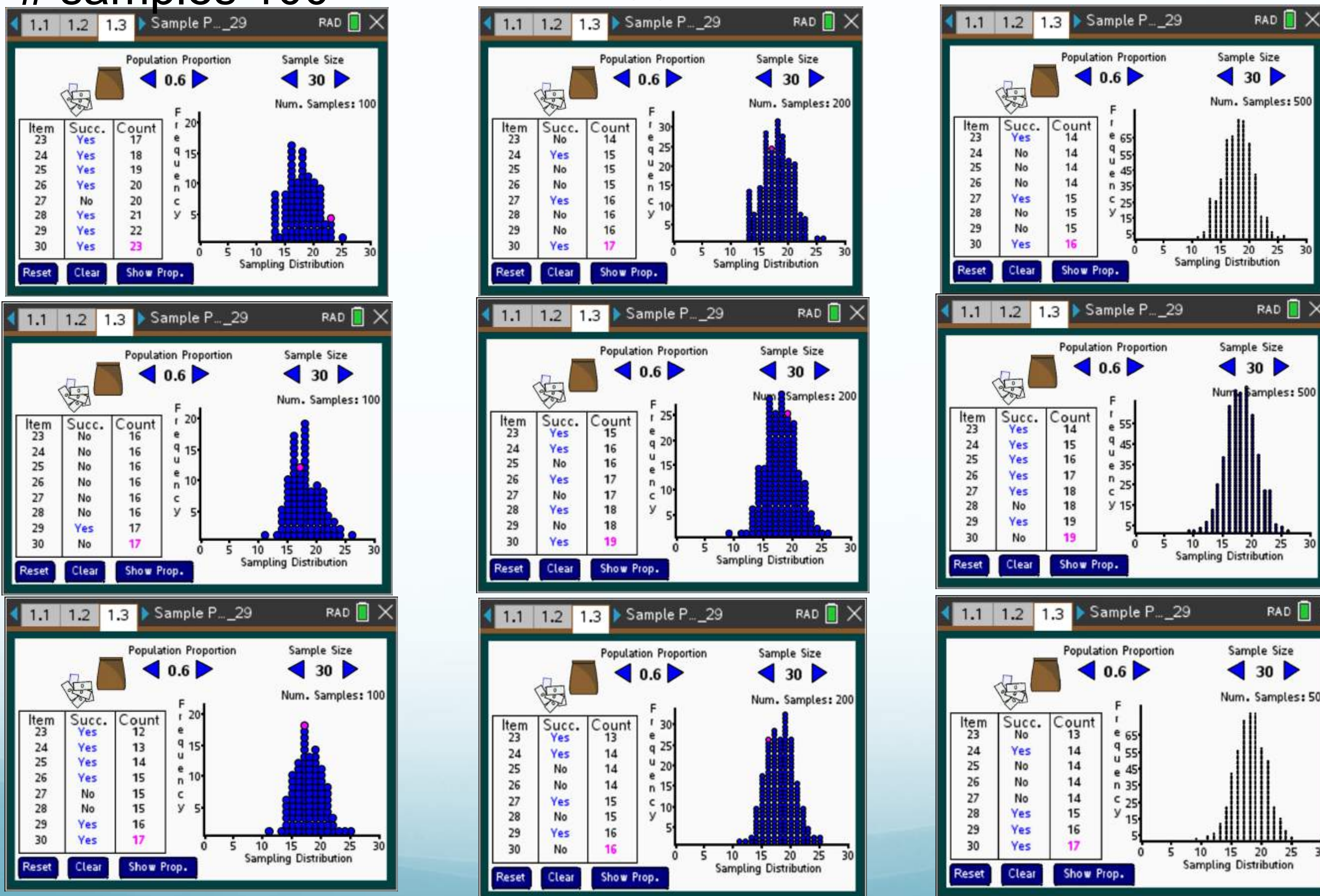
Interocular impact

p = 0.6

samples 100

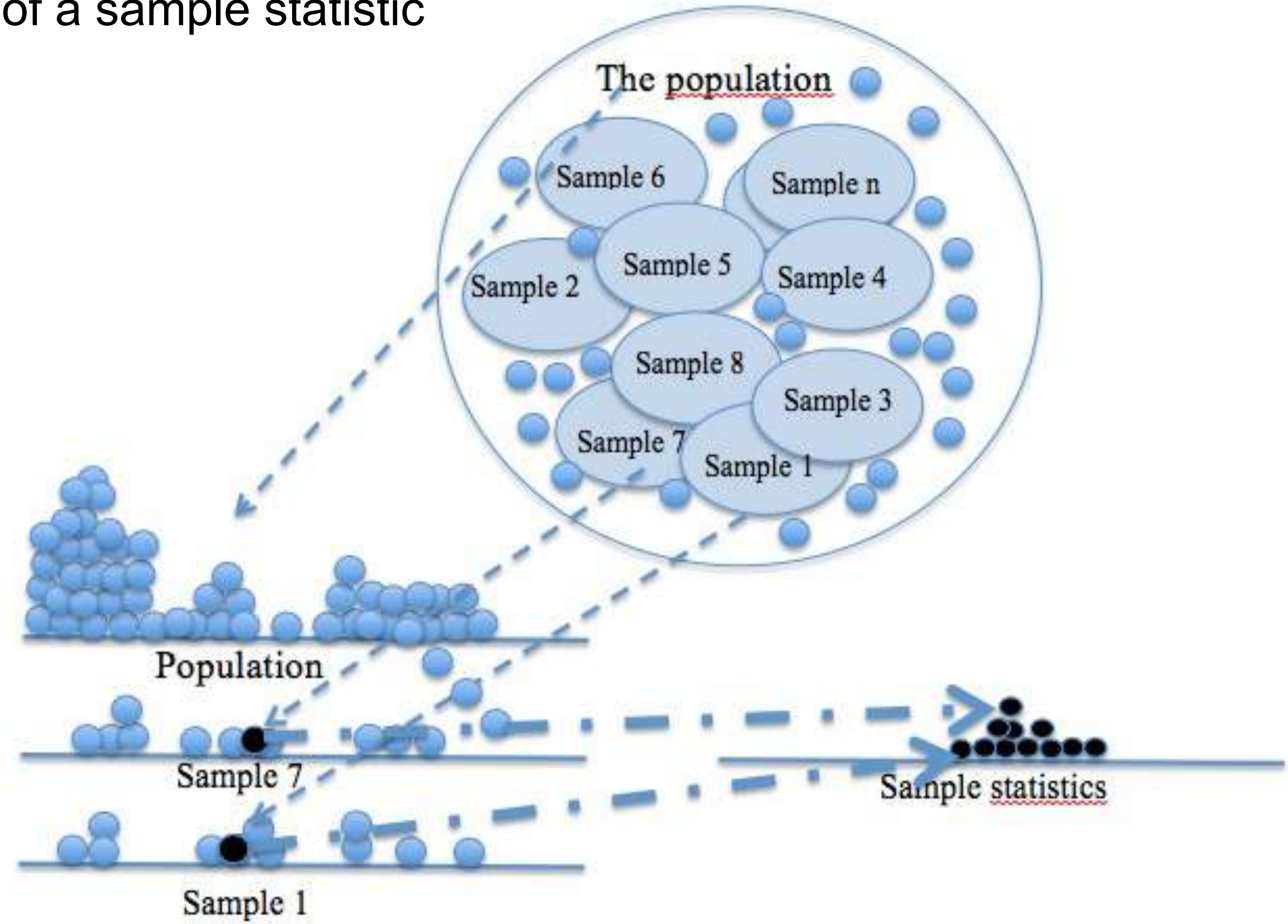
200

500



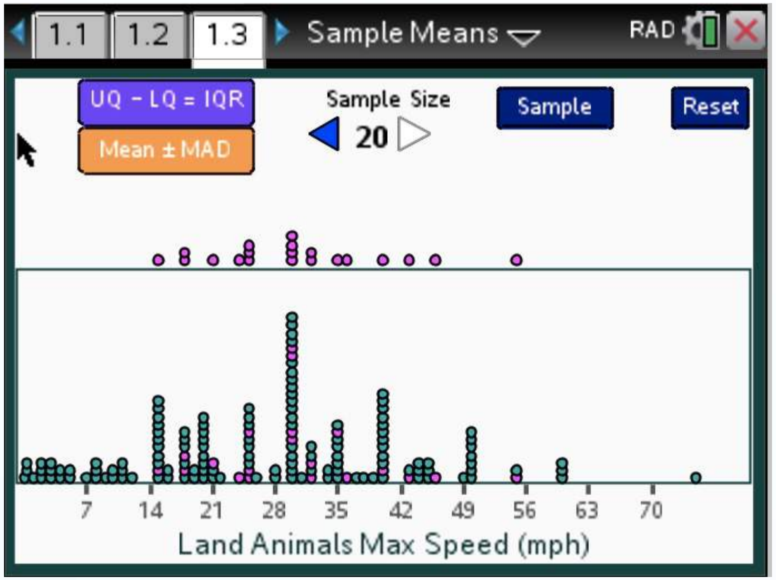
Building Concepts: Statistics and Probability, Sample proportions, 2015

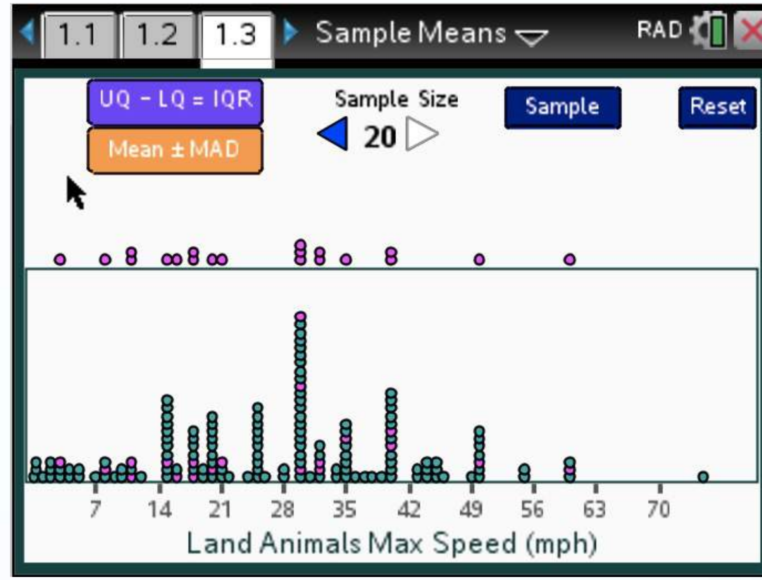
Distinguishing distributions of a population, a sample and of a sample statistic

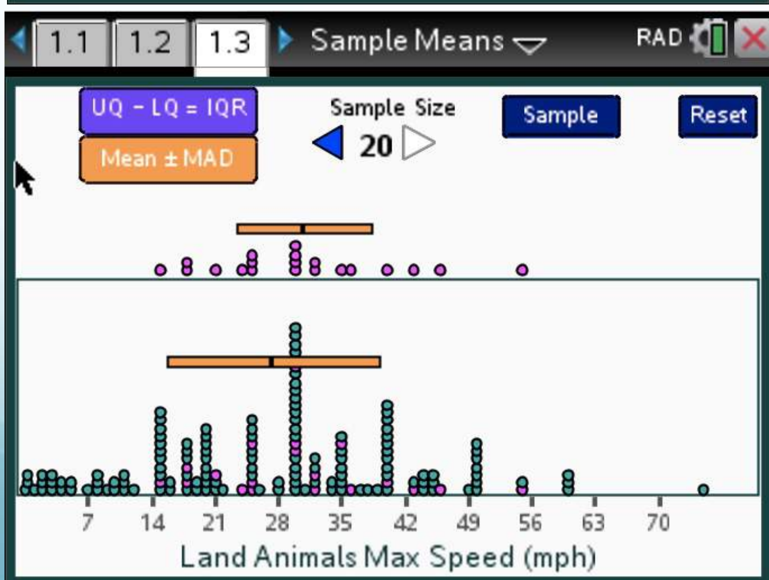


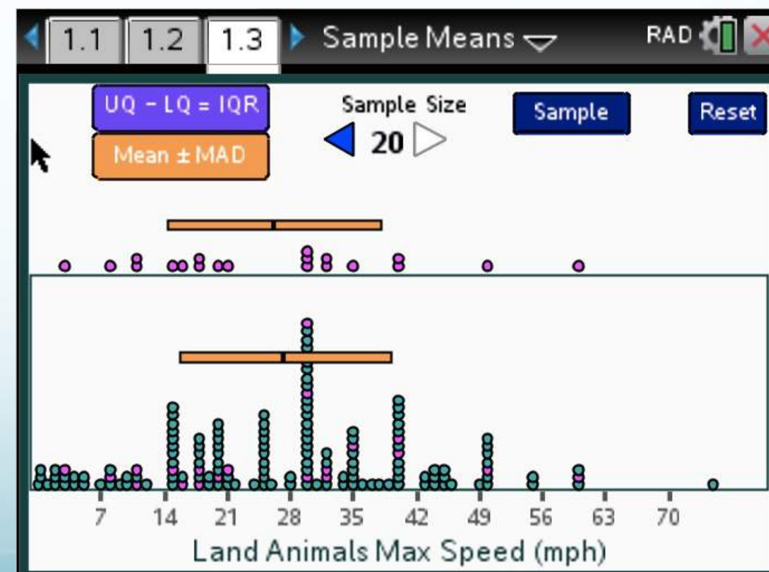
Burrill, 2019

Populations and samples

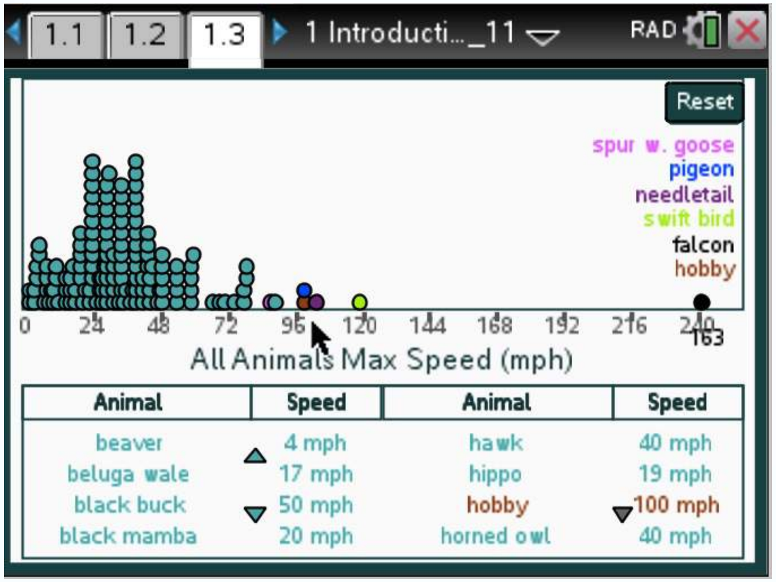






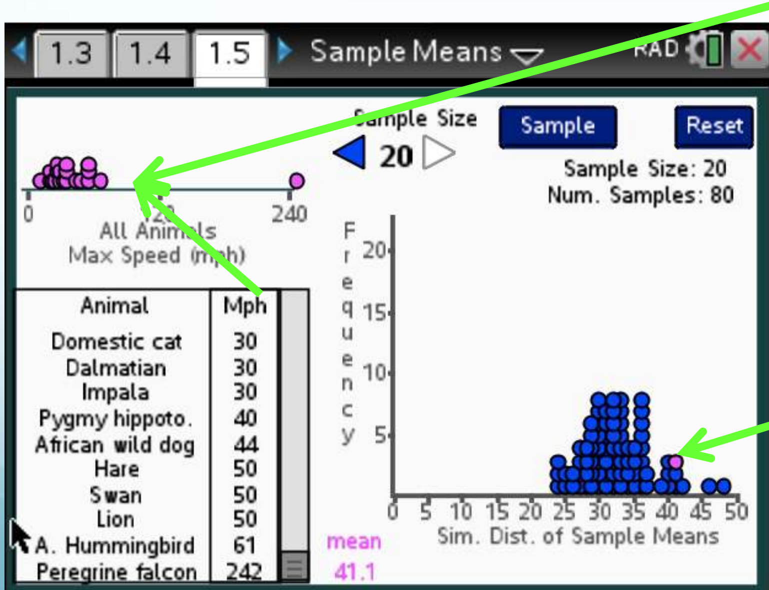


Building Concepts: Statistics and Probability, Sample means, 2015



Population: speeds of assorted animal types

Variability in sampling:

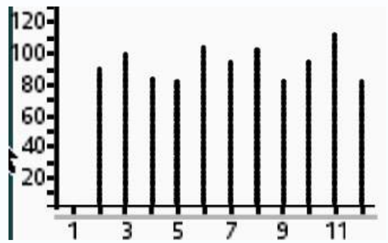


Distribution of maximum recorded speeds of 20 randomly selected animal types

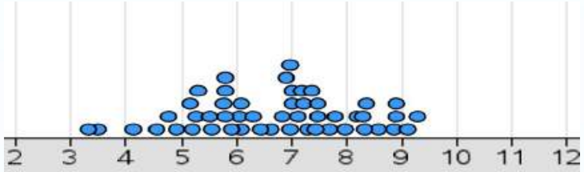
Sampling distribution of means from each sample

Distributions: population, sample and sample means

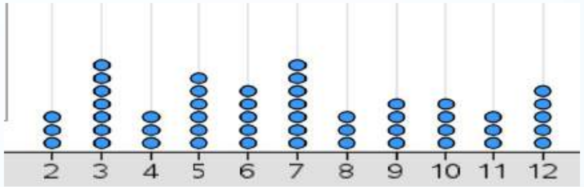
Population



Sample?

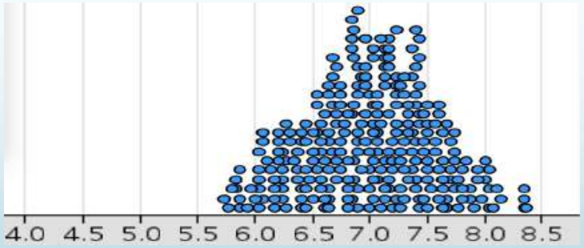


a

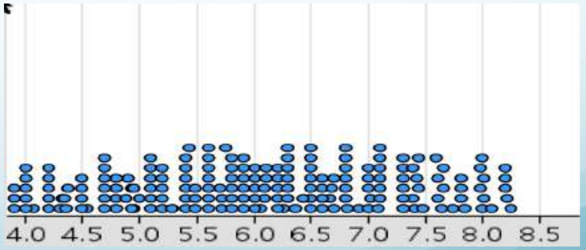


b

Sample means?



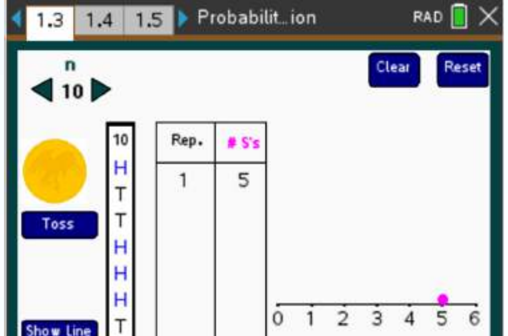
c

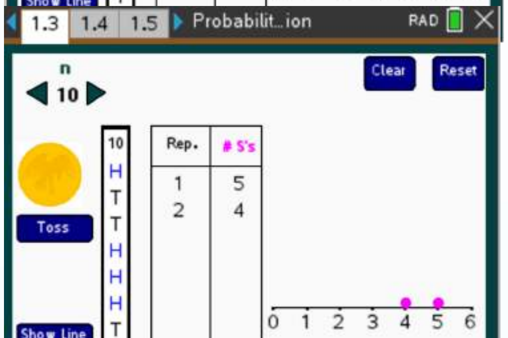


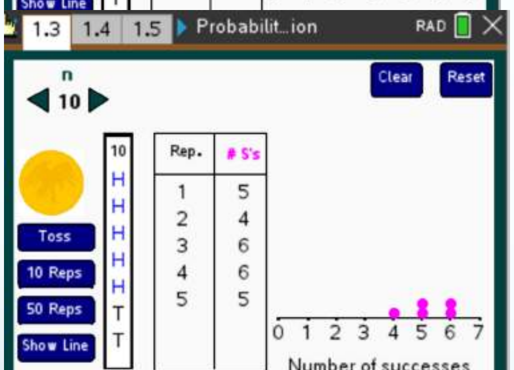
d

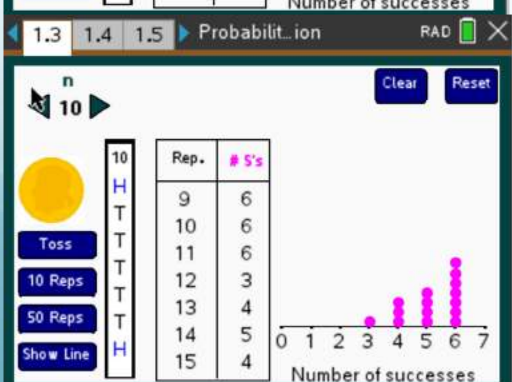
Building a distribution

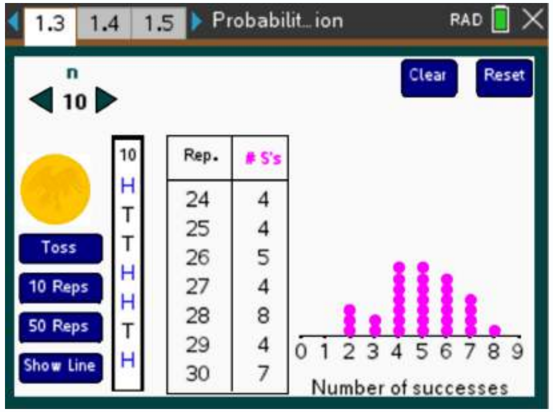
- What are the chances of passing a ten question true false test by guessing?

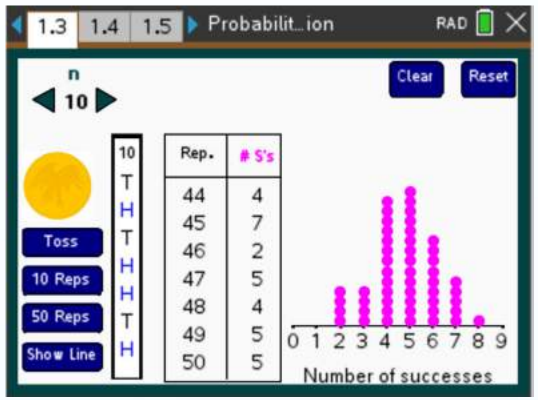


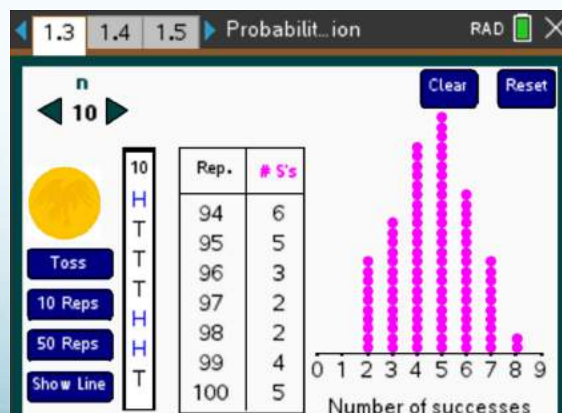


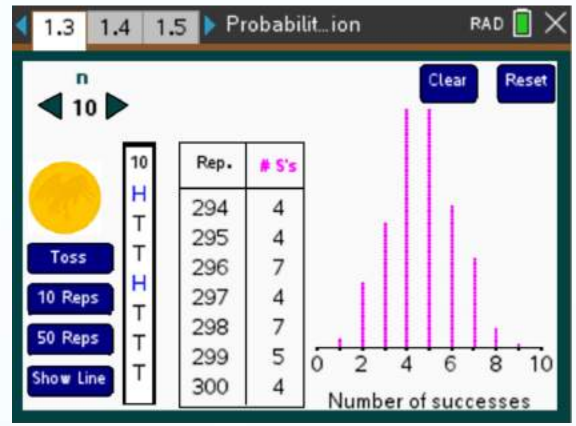


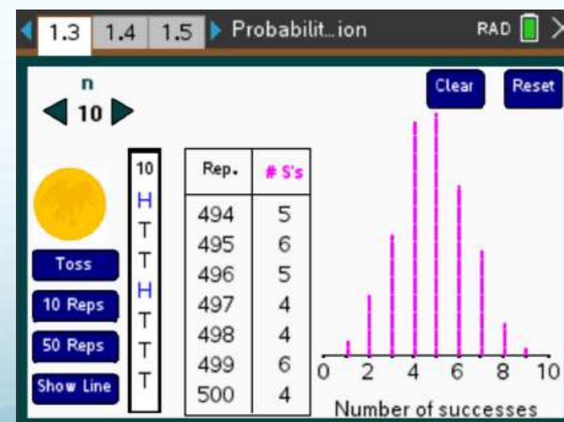












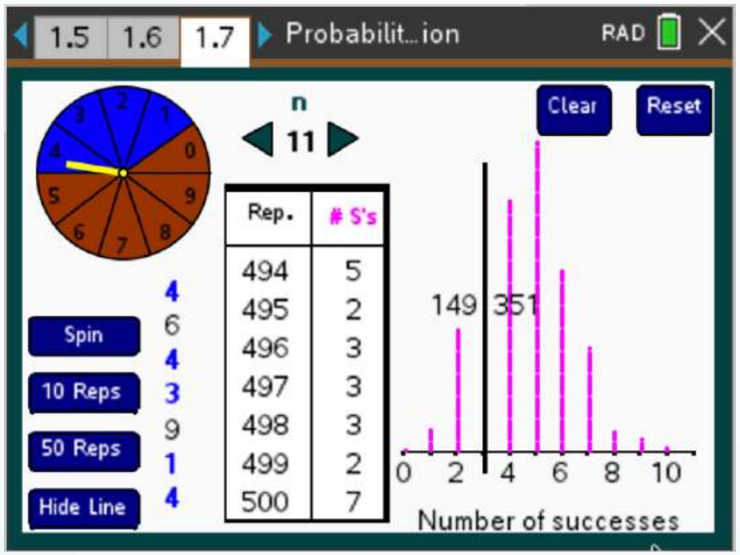
Building a sampling distribution

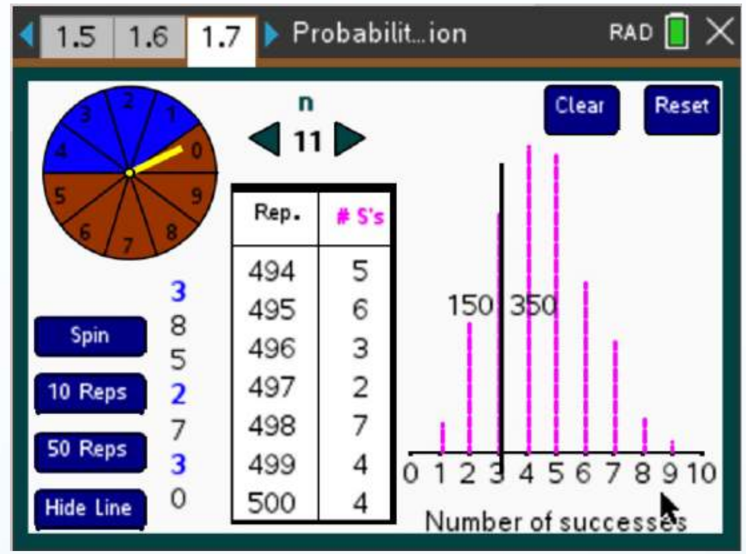
Building Concepts: Probability and Simulation, 2016

Major concern or blip? Dissecting James Harden's recent slump (Brett Koremenos, 1/29/20 https://www.yardbarker.com/nba/articles/major_concern_or_blip_dissecting_james_hardens_recent_slump/s1_13132_31170638)

In 11 games in January, 2020, Houston Rockets superstar, James Harden had a shooting average a bit over 30%, well off his season average of more than 40% from the field. Is Harden really in a slump or is it a blip?

Major concern or blip? In 11 games in January, 2020, Houston Rockets superstar, James Harden had a shooting average a bit over 30%, well off his season average of more than 40% from the field.





Is something that happens about 42% of the time by chance (149/351 or 150/350) a major concern or a blip?

Important concepts in understanding sampling distributions

- Distribution- Collection of data showing frequency of values and typically represented graphically
  - Mean as fair share and as balance point
  - Variability in distributions
    - Deviations from the mean (mean absolute deviation/ standard deviation)
- Sampling
  - Random
    - Population and representativeness
  - Variability in sampling
    - From sample to sample
    - Within samples
- Sampling distributions typically “settle down” or stabilize

Supporting the development of concept images

- What is
  - confounding?
  - rate of change?
  - co-variation?
  - z-score?
  - scope of inference?
  - correlation coefficient?
  - coefficient of determination?
- When does this concept come up? What is the concept connected to?
- How can the concept be encapsulated in a dynamic interactive visualization?
- What are some examples and non-examples?

Is what students don't understand important?

"My students struggle with the basics of confidence intervals, particularly with what will make them "wider" or "narrower" and whether "wider" or "narrower" is more desirable."

Is what students don't understand important?

"My students struggle with the basics of confidence intervals, particularly with what will make them "wider" or "narrower" and whether "wider" or "narrower" is more desirable."

This seems to go back to understanding variability and what it "looks like" graphically ...

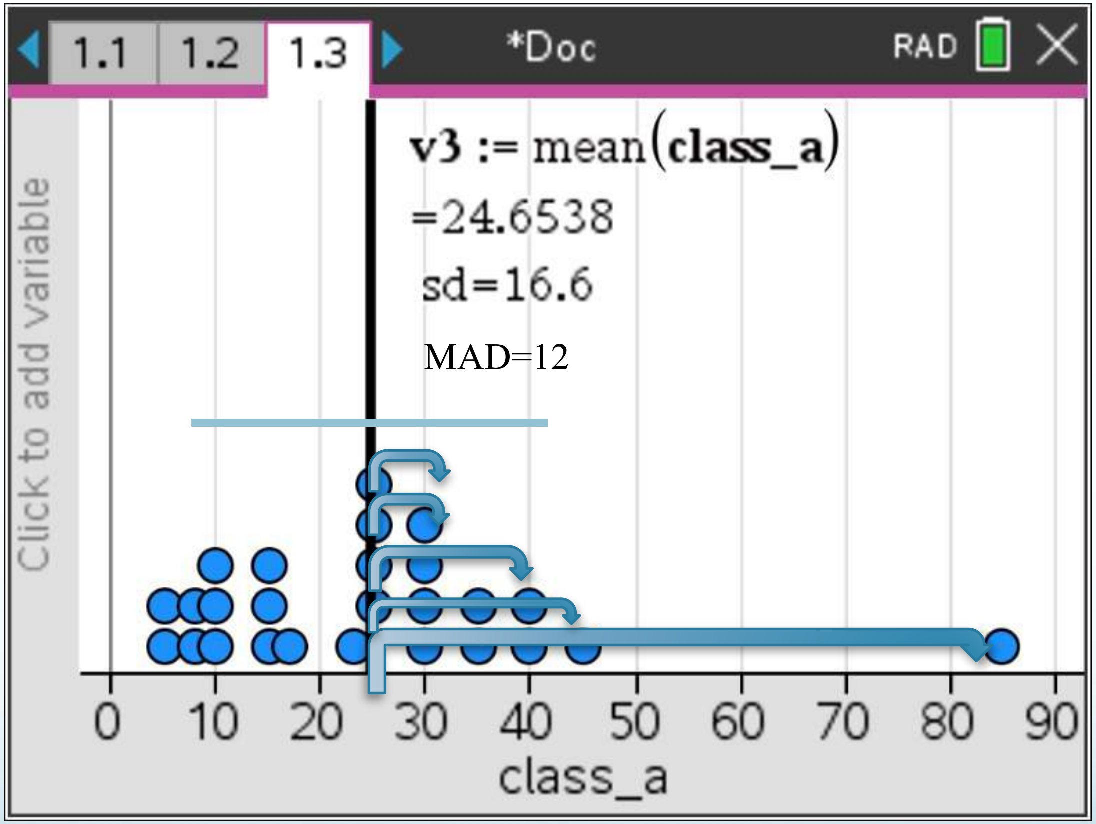
A physical model

How long did it take you to get from home to school today?

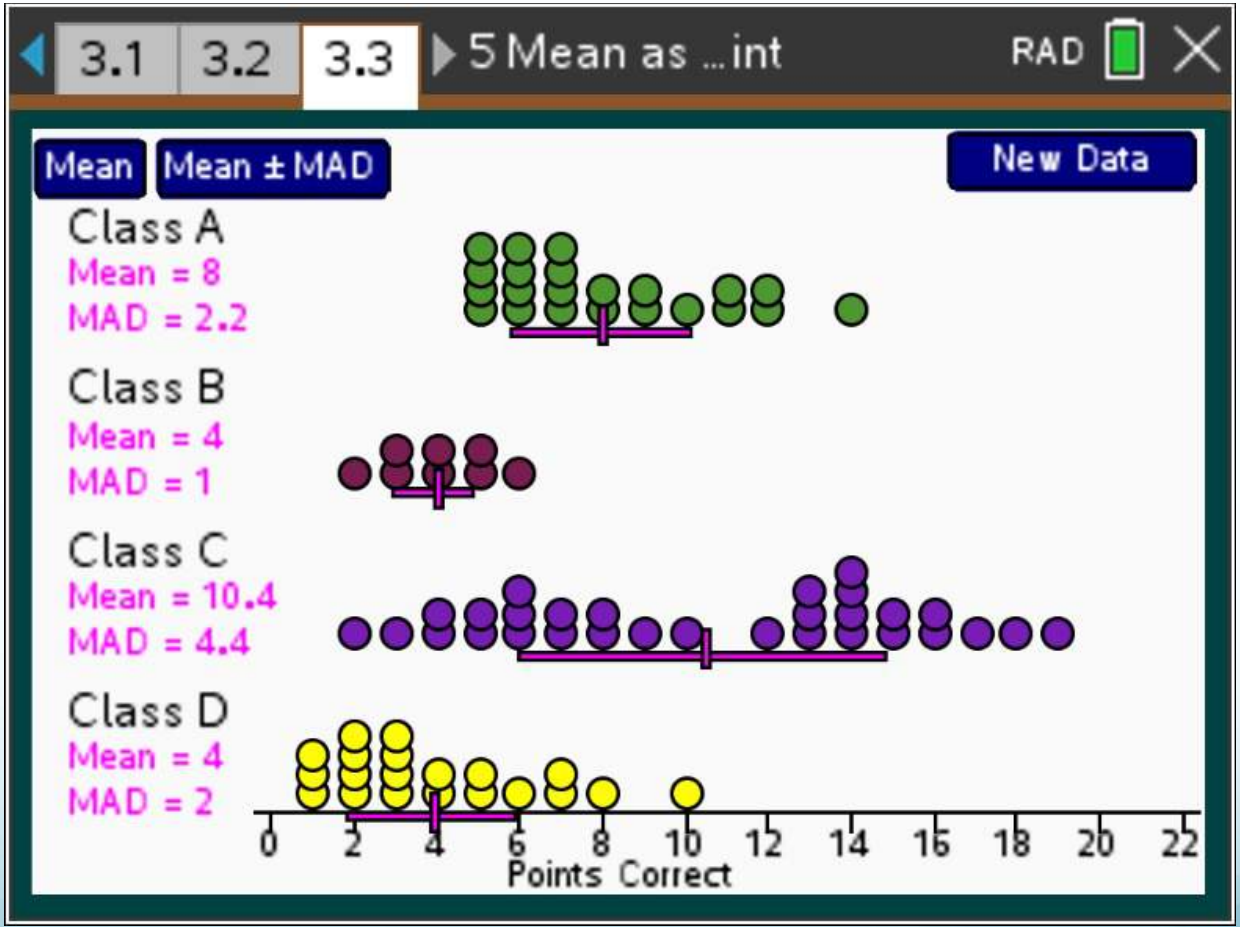
0	5, 5, 8, 8
1	0, 0, 0, 5, 5, 5, 7
2	3, 5, 5, 5, 5,
3	0, 0, 0, 0, 5, 5
4	0, 0, 5
5	
6	
7	
8	5

- Was there consistency in the time it took people to come from home to school? Why or why not?
- Describe the variability in the times it took people. Explain your reasoning.

Summary measures

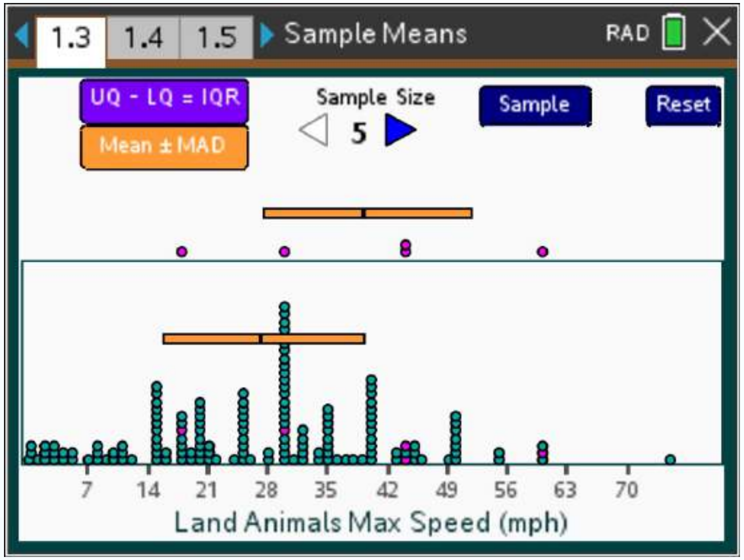


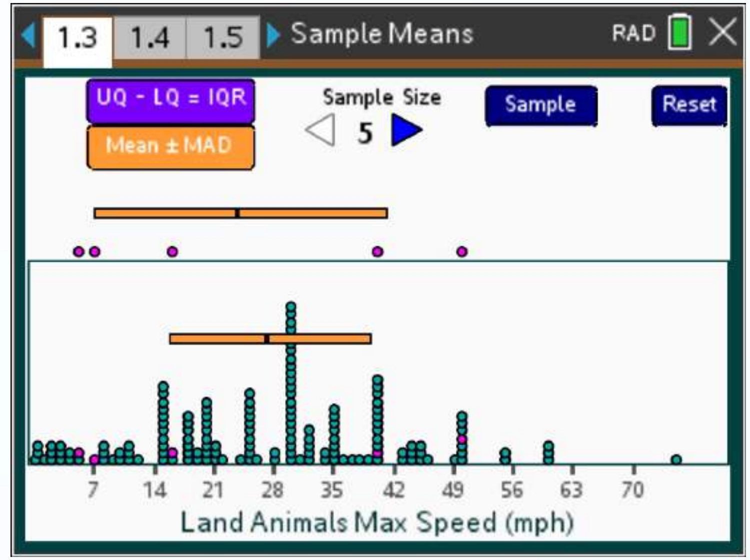
Which class had the most variability in scores? Explain your reasoning.

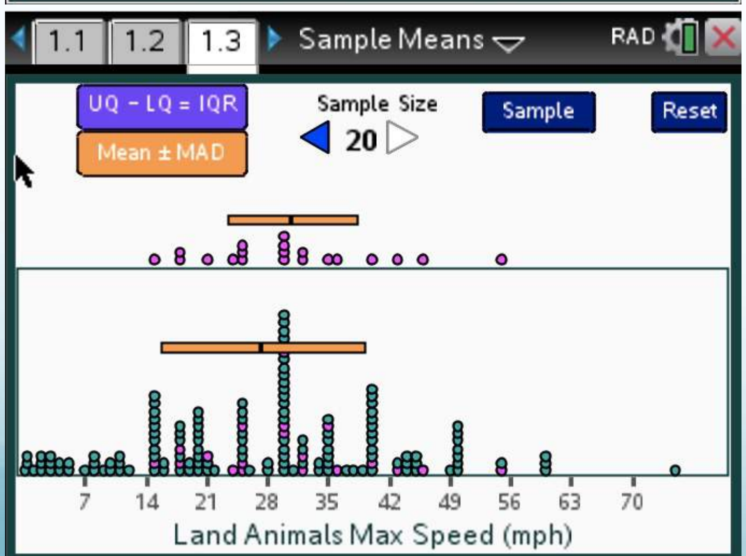


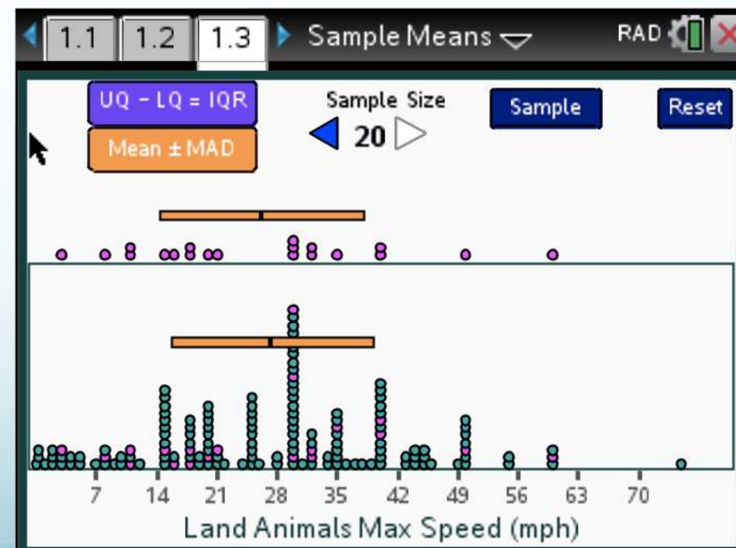
Building Concepts: Statistics and Probability, Mean as balance point, 2015

Describe the variability from sample to sample. Within each sample.





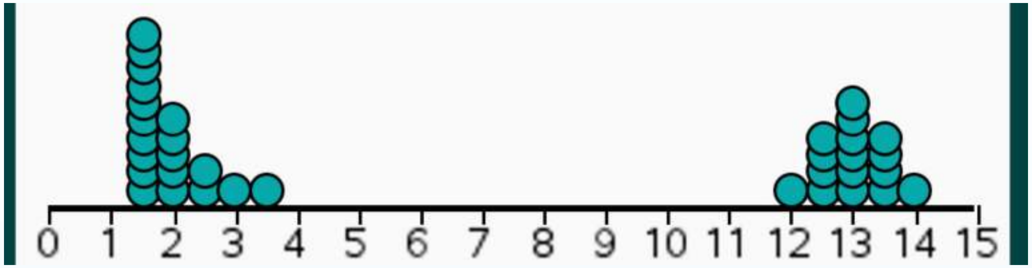


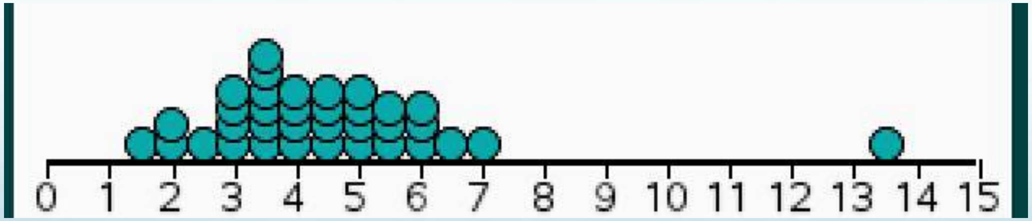


Building Concepts: Statistics and Probability, Sample means, 2015

Range as a measure of variability

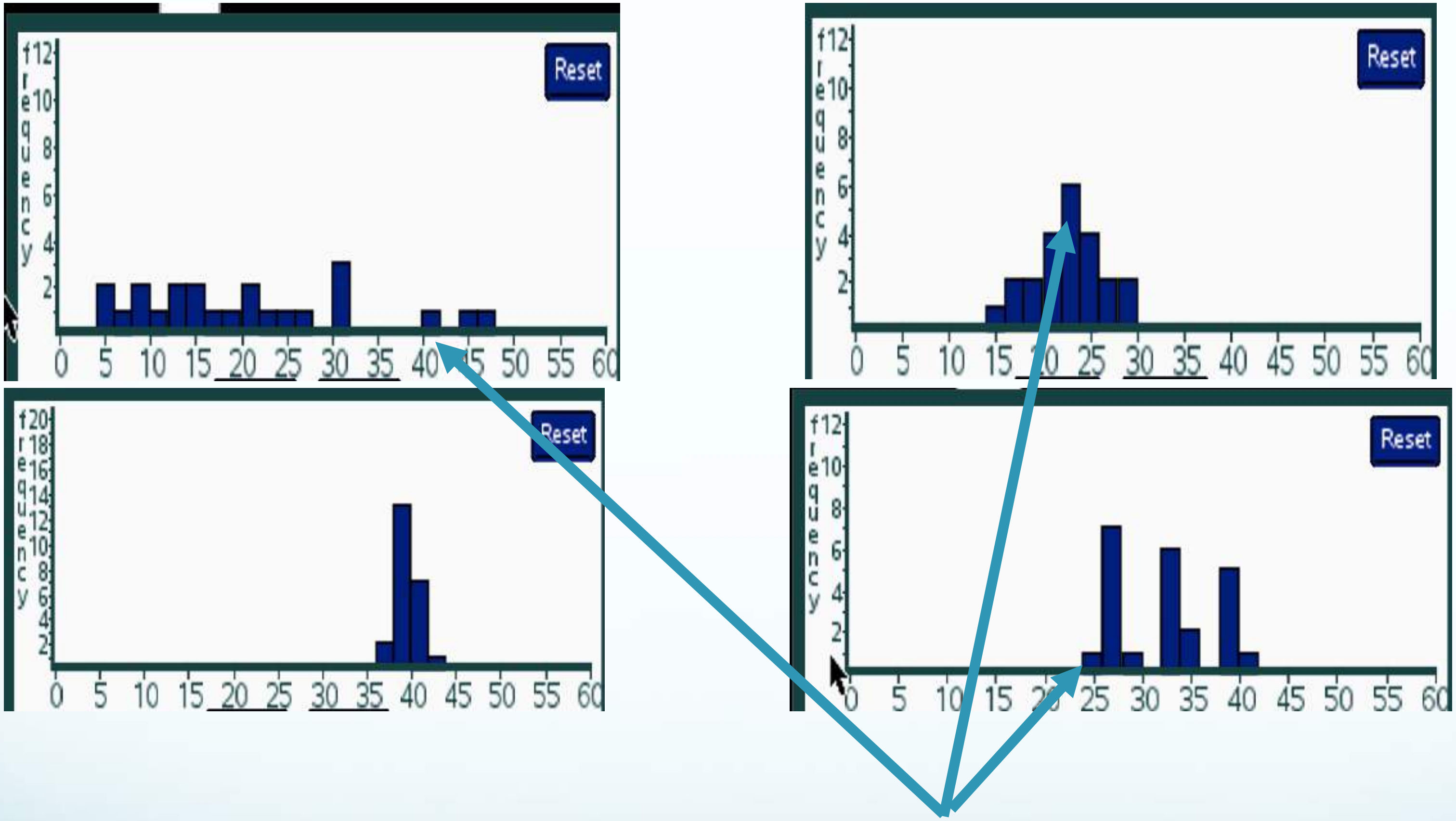
- Create a distribution that illustrates why the range is not an adequate measure of variability.
- What is useful about the range?





Hours per day online

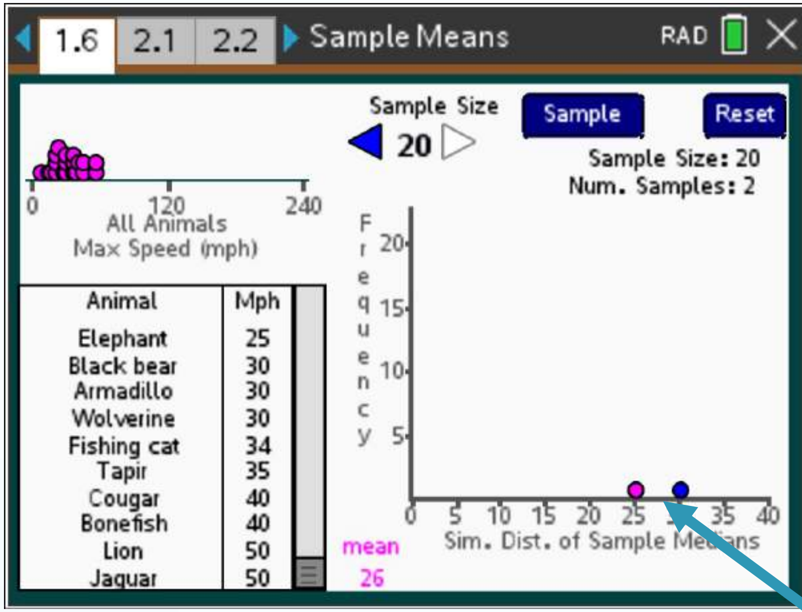
Visualizing variability

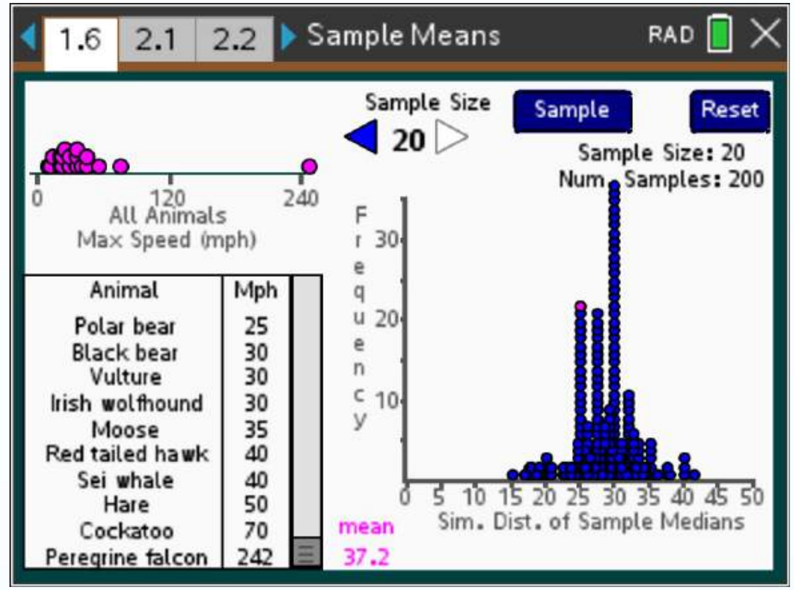


- What does each bar represent? Explain,
- Rank the distributions from least to most variability.

Justify your ranking.

Visualizing variability

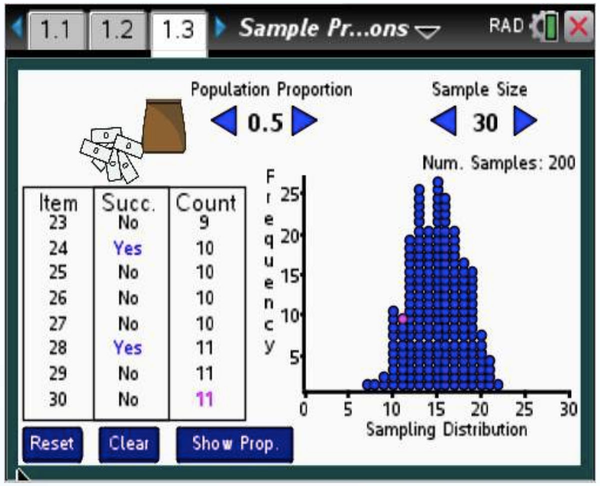


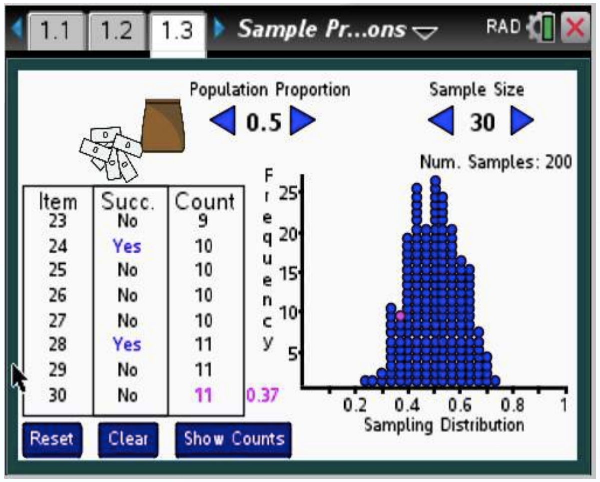


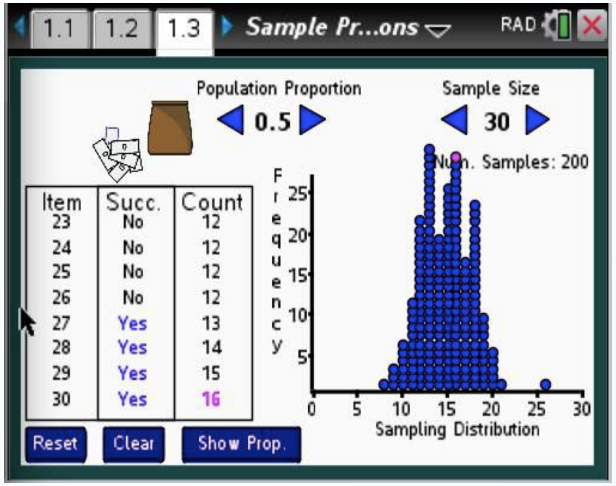
- What does each dot represent? Explain.
- What does the collection of dots represent? Estimate the mean and the standard deviation.

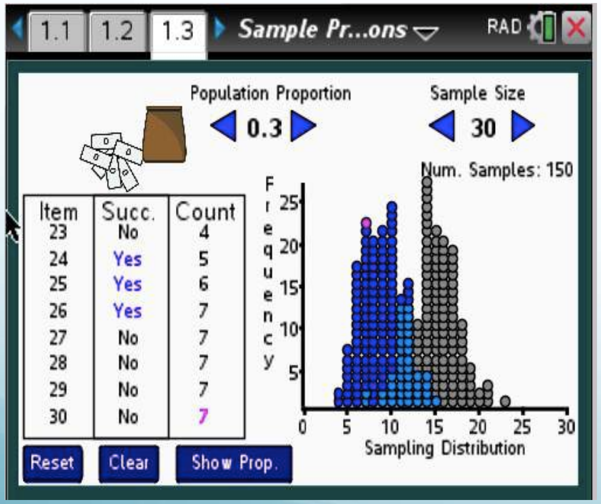
Building Concepts: Statistics and Probability, Sample means, 2015

To inference: Random samples when the population proportion is known



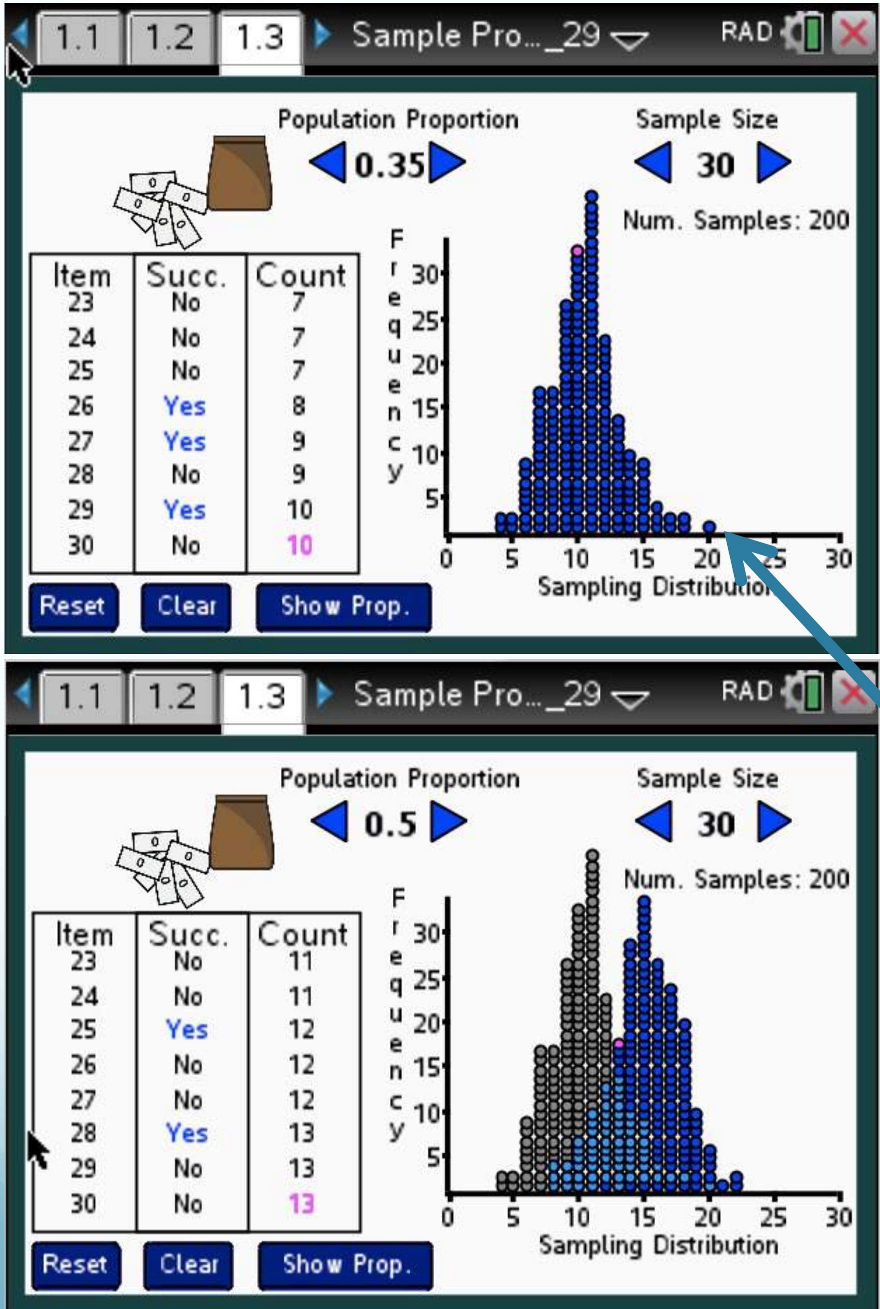






Understanding the known:

- Is an outcome of 5 yesses in a random sample of 30 likely? Explain. An outcome of 25? Of 18? Of 0.7?
- Is an outcome of 12 plausible for both a population proportion of 50% and 30%? Explain.



Chance variability: simulated sampling distributions for given population proportions and sample sizes

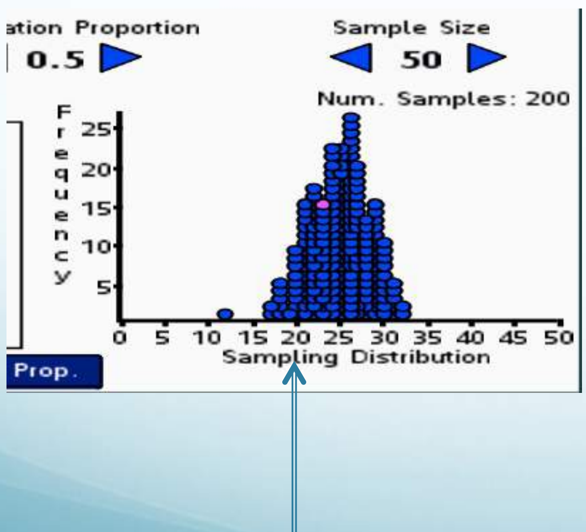
How likely is an outcome of 20 successes in a sample of size 30 for a known population proportion of 0.35?

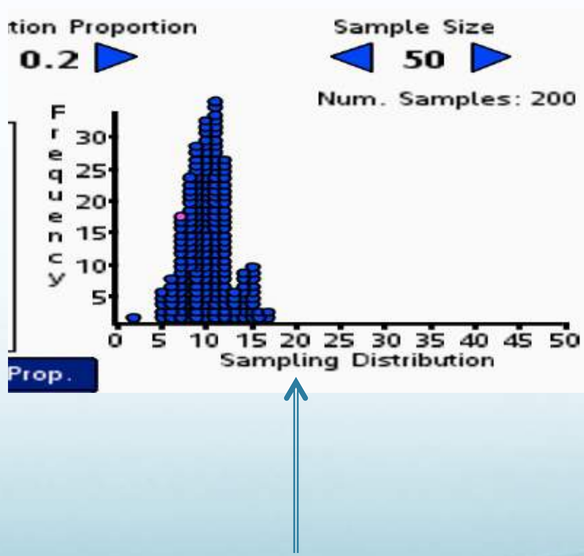
Building Concepts: Statistics and Probability, Sample Proportions, 2015

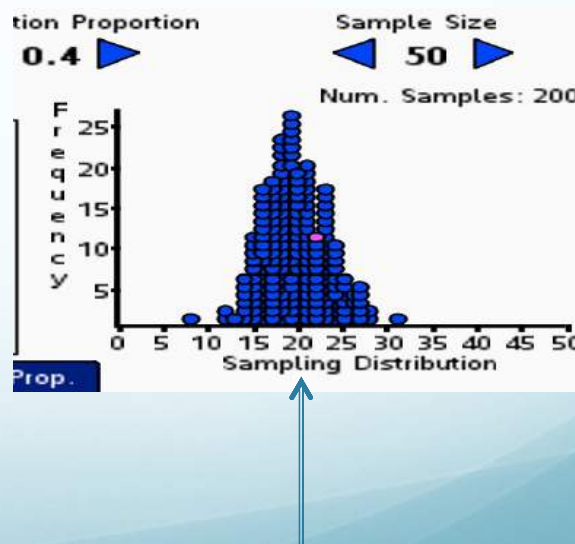
Using the known to estimate the unknown

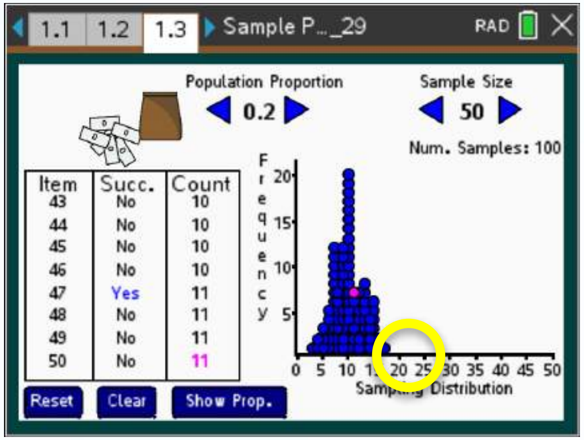
A sample of 50 M&M's found 20 that were brown. The distributions below represent simulated sampling distributions of the proportion of brown M&M's for populations where 40%, 20% and 50% were brown. Which of the populations is plausible for the sample? Explain your reasoning.

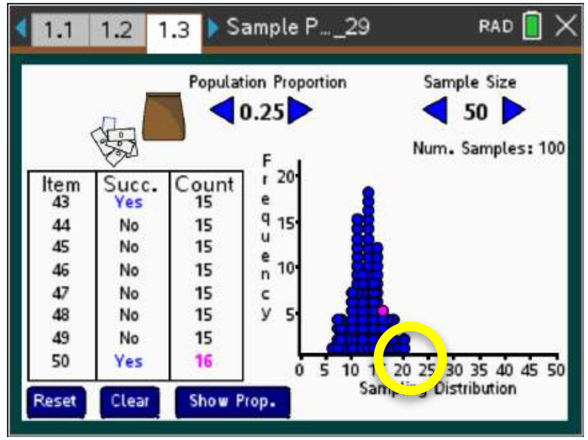
a) Pop 50% brown b) pop 20% brown c) pop 40% brown

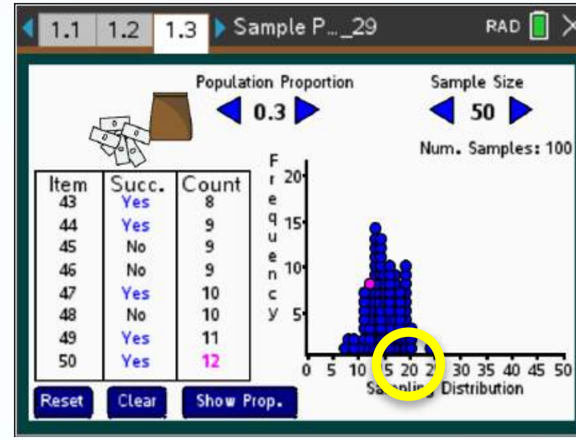


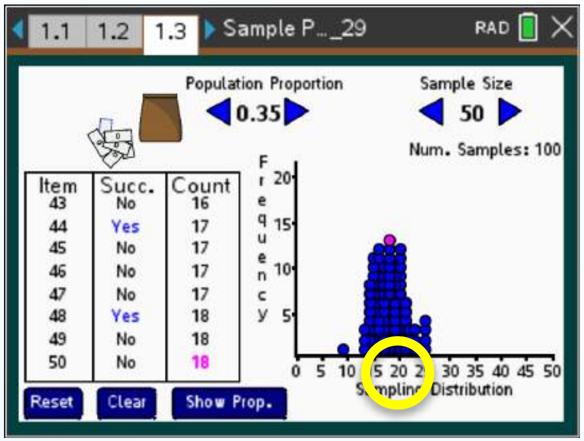


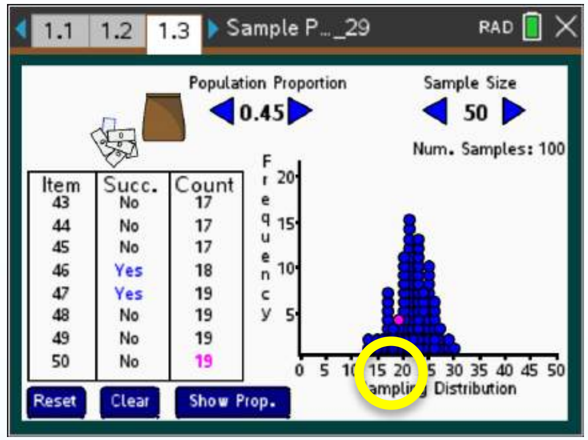


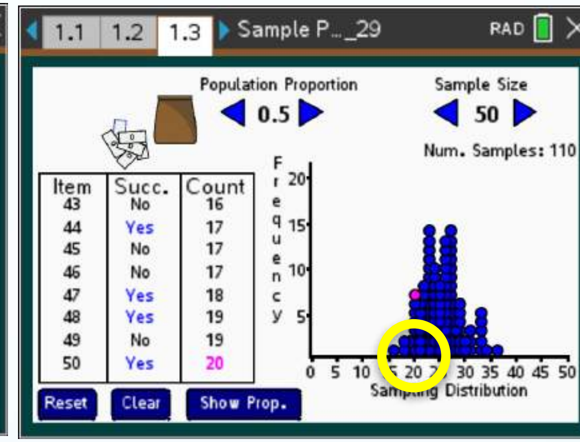


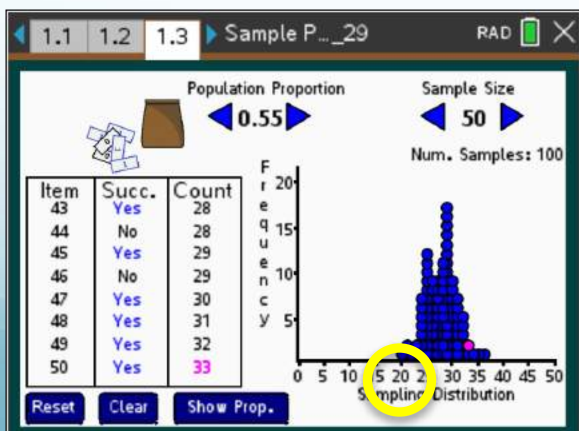


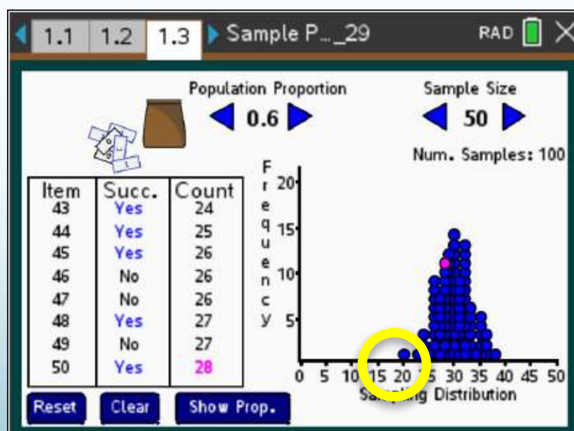










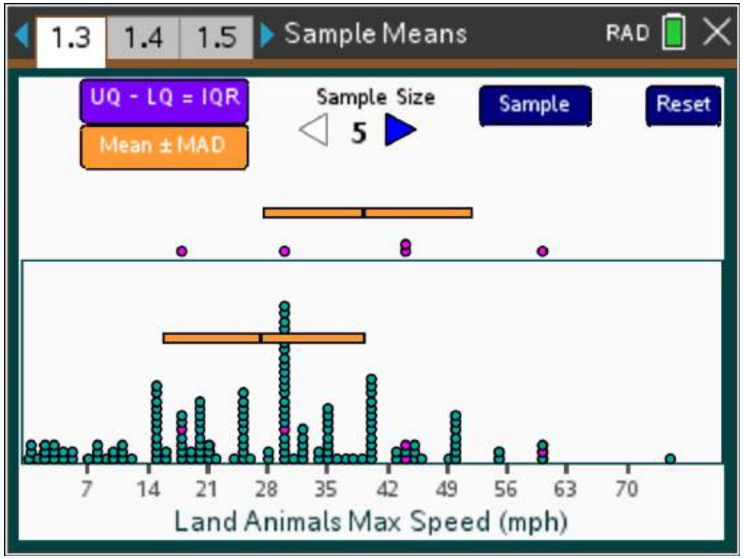


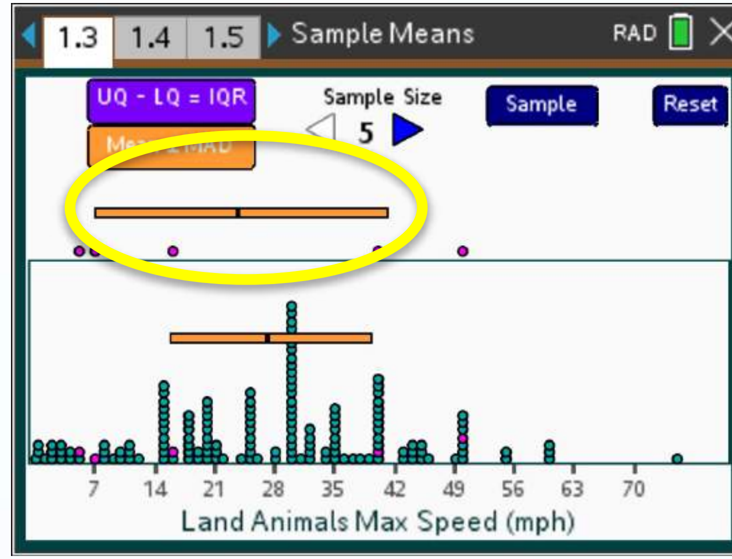
Which are plausible populations for a random sample with 20 yesses?

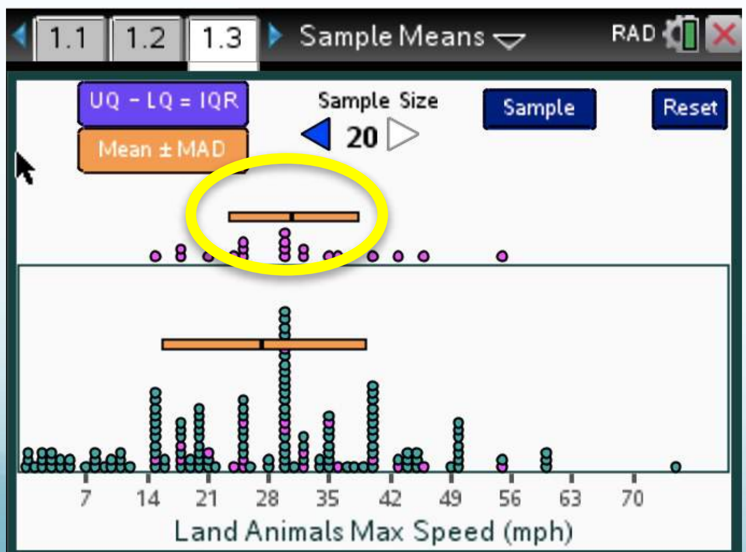
From the known to the unknown

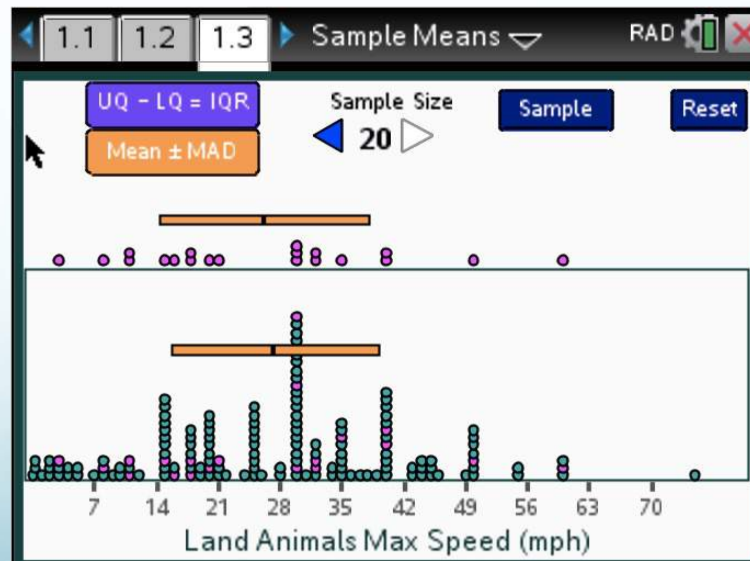
- Identifying all plausible population proportions for the observed outcome of 20 brown in a sample of 50 leads to an interval – say 0.25 to 0.55 (or 0.24 to 0.56 if you use 2 sds).
- What will happen if the sample size increases?

Revisiting populations and samples



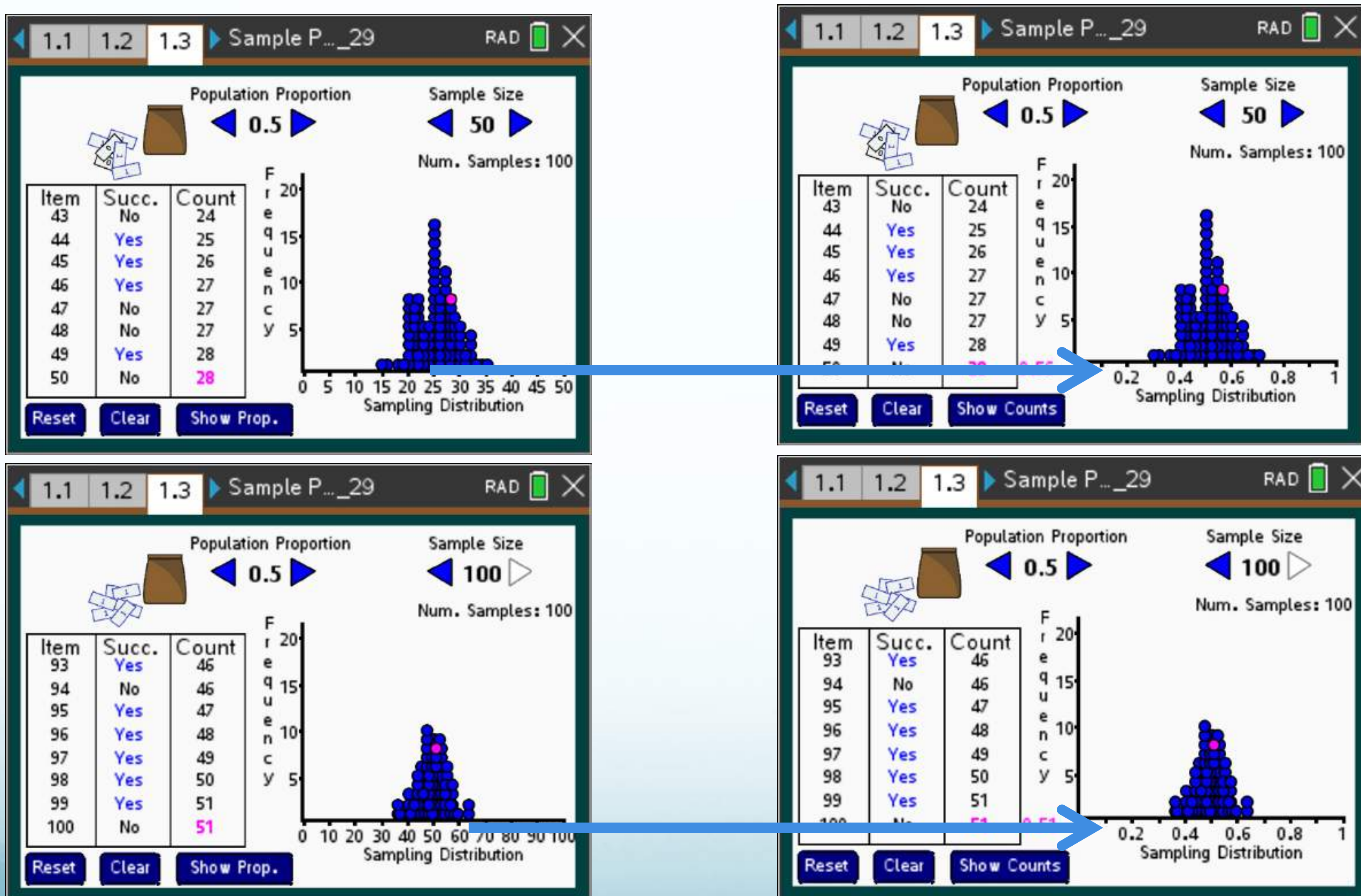






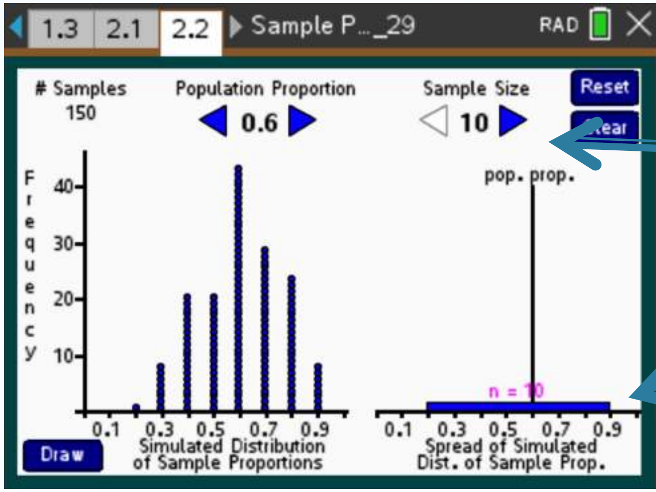
Building Concepts: Statistics and Probability, Sample means, 2015

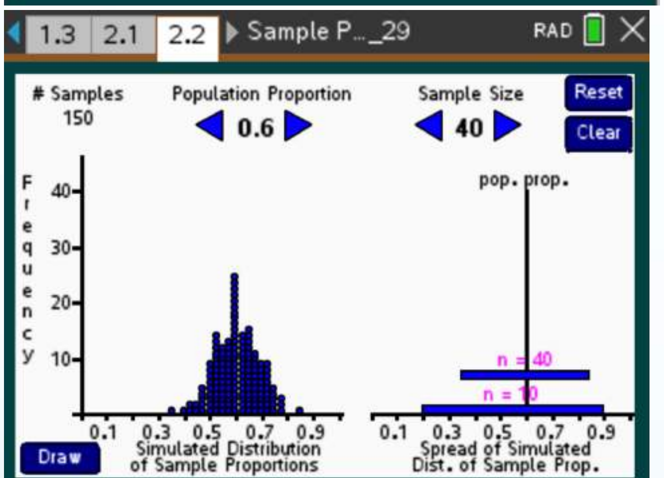
Counts vs proportions

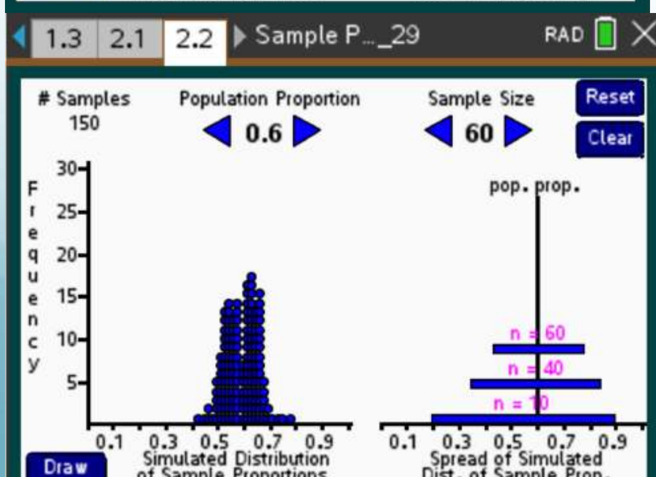


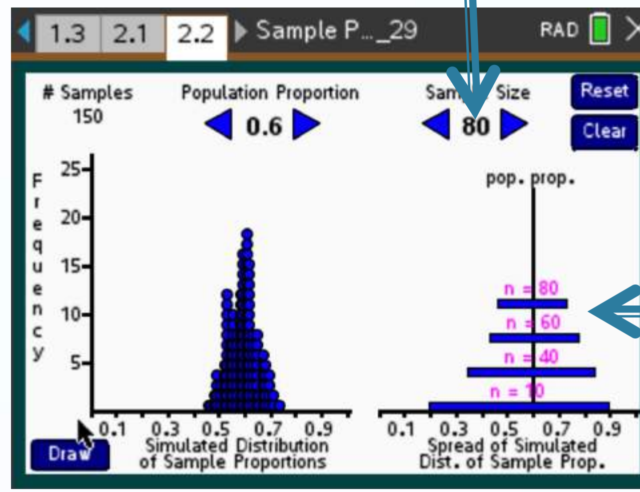
Building Concepts: Statistics and Probability, Sample proportions, 2015

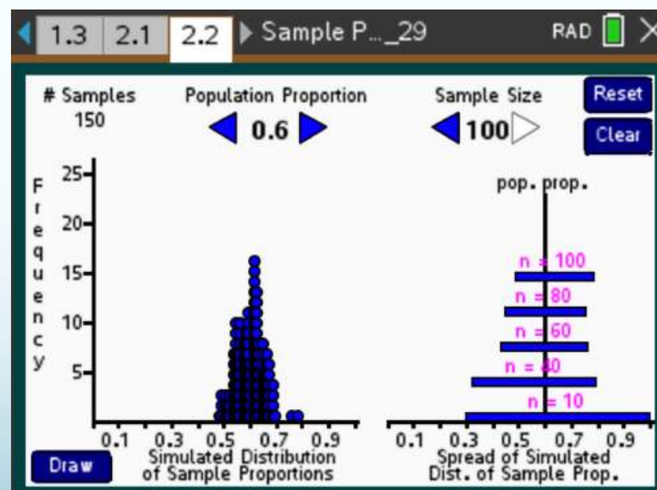
Sample size and variability







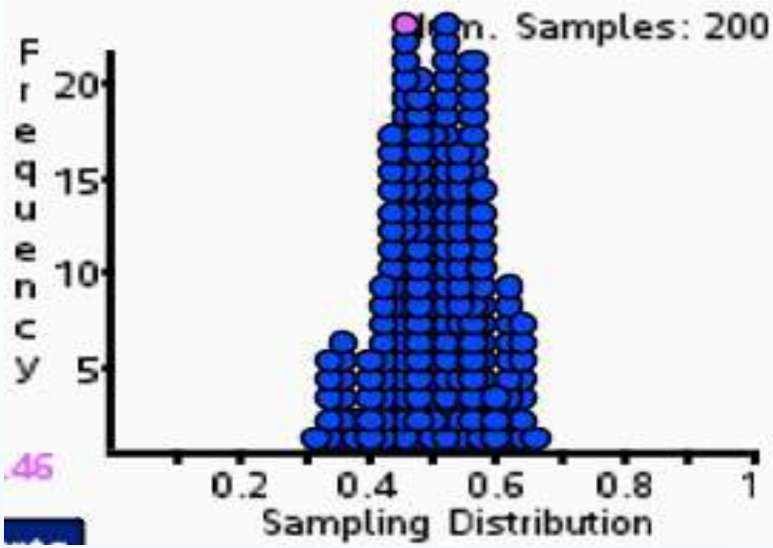




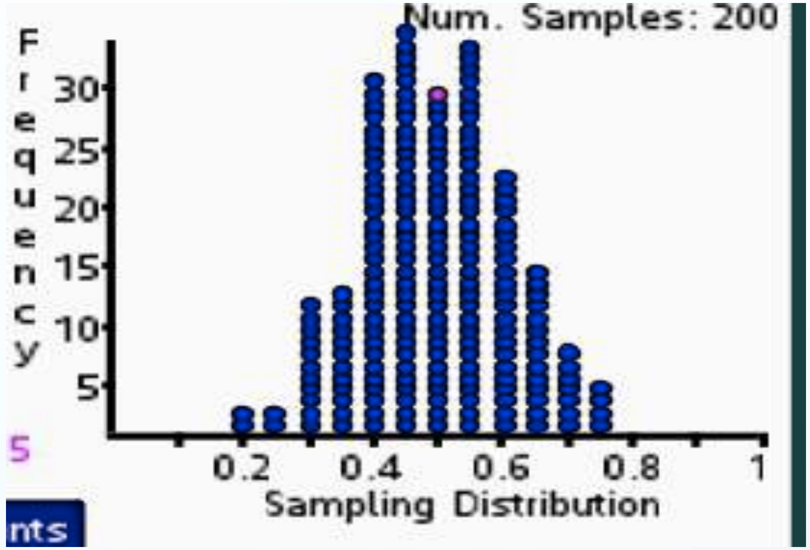
Building Concepts: Statistics and Probability, Sample Proportions, 2015

Changing sample size

a)



b)



Which simulated distribution of sample proportions came from a sample of size 30 and which one from a sample of size 50? Explain your thinking.

- “My students struggle with the basics of confidence intervals, particularly with what will make them "wider" or "narrower" and whether "wider" or "narrower" is more desirable.”
- If the sample size is 100, will the interval of population proportions that could plausibly have produced a sample of 0.4 brown M&Ms, 0.25 to 0.55, increase or decrease? Explain your thinking.

Misunderstandings

- Believe outcomes that "look more random" have a greater probability of occurring as those that don't (Lefebvre, 2010).
- Assume two samples from the same population will be similar (Tversky & Kahneman, 1971)
- Believe sampling distribution for a sample statistic will look like that of the population (for $n > 1$)
- Assume sampling distributions for small and large sample sizes have the same variability
- Don't consider the variability across all possible samples, and how their sample might fit into that range of possibilities (Chance, delMas, & Garfield 2004)
- Confuse the distribution of a population, a sample from population, and the sampling distribution of a sample statistic (Wild, 2006).
- Believe the width of a confidence interval increases with sample size
- Confuse confidence intervals and levels (Finch & Cumming, 2008)

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Scope of inference

“Regarding concepts that students struggle with, I'd really like to have better ways of visually summarizing how to identify/discuss the scope appropriate for a given research study (generalizability and/or causality, based on the kinds of randomization used in collecting the data). I use summary tables to organize this information for students, but both of these topics can be a persistent source of difficulty for many students.”

Developing concept images

- A concept image is not usually built on definitions but essentially determined by **typical examples** (Vinner & Dreyfus, 1989)
- The concept **definition does not seem to play any role** when students are working on problems (Vinner, 1994)
- **Explanations for concepts will easily be forgotten** if students are not able to develop own ideas and associations (Rösken & Rolka, 2007).
- Students' actions should be repeated with provisions for feedback.
- Students should repeat these actions in structurally similar problems in a variety of contexts to develop a robust abstraction of the concept. (Oehrtman, 2008).

Scope of inference: Examples and non-examples

- Random rectangles (Schaeffer et al, 1996), random circles, etc. (human judgment is typically biased)
- Each student (or group of students) should ask a question – e.g., the number of hours per day students exercise; ask athletes and nonathletes and then a random sample
- Observe a random sample of people in a coffee shop in the morning and in the afternoon. Describe the difference.
- Design an experiment that does not have random assignment and carry it out. Contrast the results with an experiment in the same context with random assignment.

Revisit the concept

- Find articles where the samples are not random; discuss the consequences
- Find situations where random assignment did not take place (early medical research all on men, Japanese men, ...)
- Talk about the COVID-19 vaccines- was there random assignment? How do you know? Why does this make a difference? Why did children under the age of 16 have to wait for the vaccine?
- ...

Bootstrap and randomization

(Pfannkuch et al, 2013; Tintle et al, 2012; Tintle et al., 2018)

StatKey to accompany *Statistics: Unlocking the Power of Data*
by Lock, Lock, Lock, Lock, and Lock

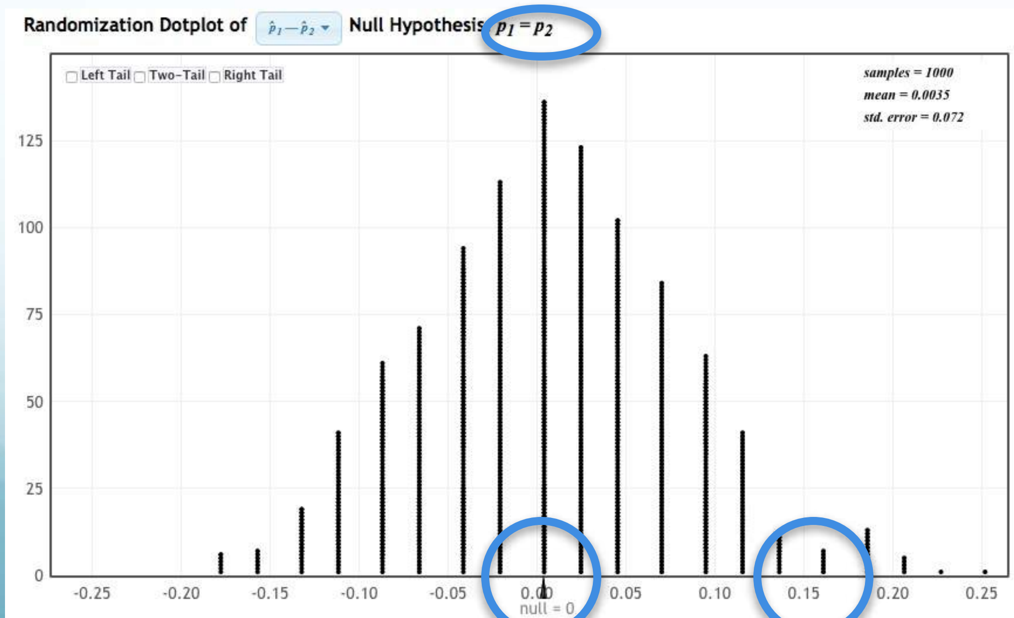
Descriptive Statistics and Graphs	Bootstrap Confidence Intervals	Randomization Hypothesis Tests
One Quantitative Variable	CI for Single Mean, Median, St.Dev.	Test for Single Mean
One Categorical Variable	CI for Single Proportion	Test for Single Proportion
One Quantitative and One Categorical Variable	CI for Difference In Means	Test for Difference in Means
Two Categorical Variables	CI for Difference In Proportions	Test for Difference In Proportions
Two Quantitative Variables	CI for Slope, Correlation	Test for Slope, Correlation

Sampling Distributions	Mean	Proportion

Theoretical Distributions	Normal	t	χ^2	F

More Advanced Randomization Tests	χ^2 Goodness-of-Fit	χ^2 Test for Association	ANOVA for Difference in Means	ANOVA for Regression

- Carpal tunnel syndrome can be treated with surgery or less invasive wrist splints. A study of **176** patients found that among the **half** that had surgery, **80%** showed improvement after three months. Only **54%** of those who use the wrist splints improved. Is there evidence of a real difference between the two proportions or could the difference have occurred by chance? Why or why not?(Song, 2002)



Original Sample

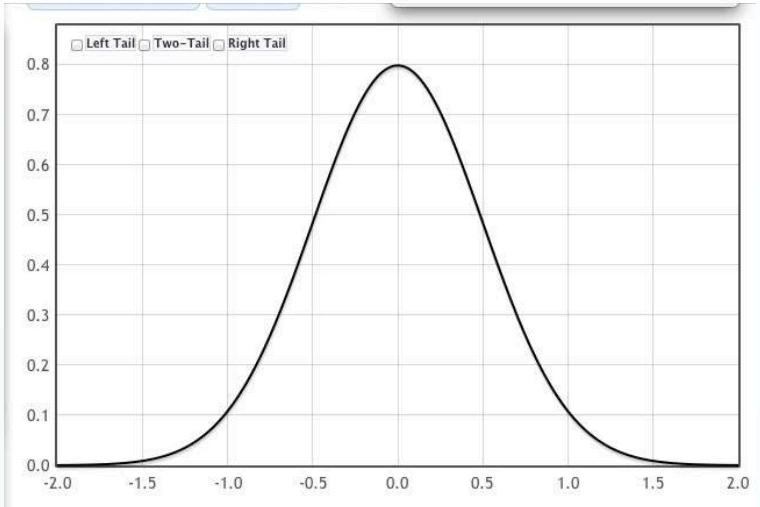
Group	Count	Sample Size	Proportion
Group 1	70	88	0.795
Group 2	48	88	0.545
Group 1-Group 2	22	n/a	[illegible]

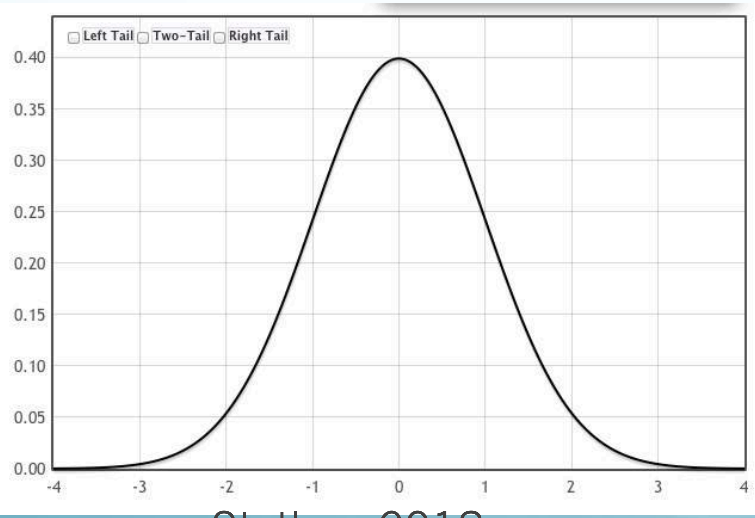
Randomization Sample

Group	Count	Sample Size	Proportion
Group 1	61	88	0.693
Group 2	57	88	0.648
Group 1-Group 2	4	n/a	0.045

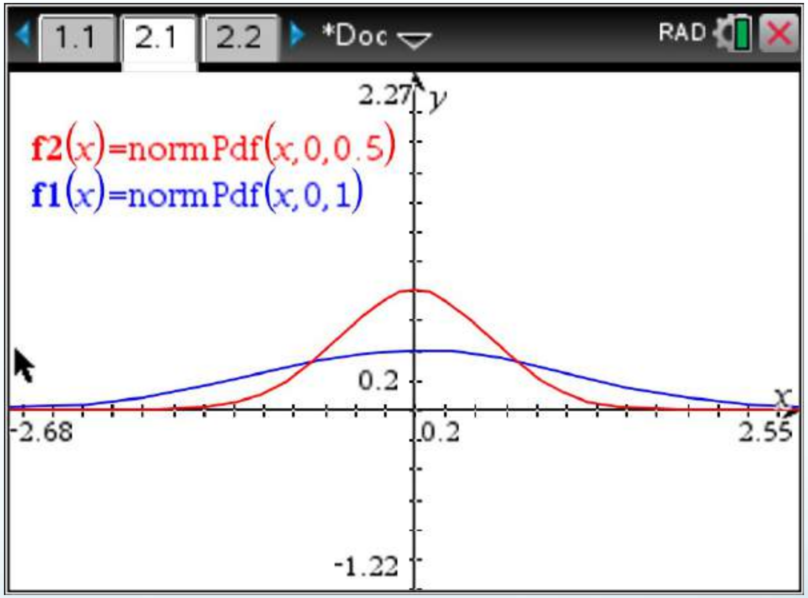
Statkey, 2018

And the difference is…?





Statkey, 2018



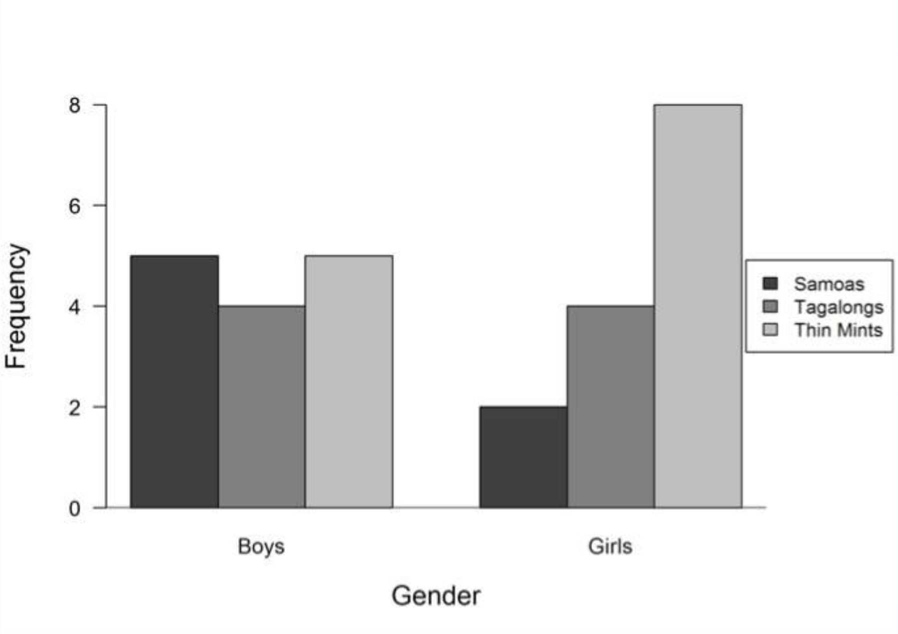
TI Nspire

Ways of knowing

- What is true and what you are trying to estimate
- What you cannot know
- Using what is true to create likely or plausible estimates

Observations about course for elementary preservice students using the applets

- Students work best in visibly random groups of three when in person.
- Better understanding of what was expected when take time to explore difference/similarities between math and statistics
- Weekly homework consists of four or five short open response questions asking for results and interpretations; not graded and quick feedback - often in the form of "why am I worried when I see ….."
- No text book but each "unit" was supported by online resources such as Kahn Academy, Penn State online, …
- Frequent quick polls or quizzes that can be analyzed at a glance



Samoas, Tagalongs, and Thin Mints are types of Girl Scout cookies. Students in a sixth-grade class surveyed their classmates for which of the three types of cookies they preferred. (LOCUS)

Were girls or boys more variable in their choices for favorite cookie? Justify your response.

(possible misconception- lose reference to context and choose the tallest bar)

Strategies to Help Students Access Fundamental Statistical Concepts

- Provide opportunities for students to develop concept images
- Focus on only one or two ideas-take time to develop the idea
- Give students experience with hands-on/concrete activities before theory or technology
- Use simulations to build understanding and explore ideas
- Emphasize statistical literacy and sense making
- Provide opportunities for students to confront misconceptions/misunderstandings

- Choose tasks that engage students in applying ideas in context
- Use words not symbols
- Avoid formulas until students have enough experience to make sense of them
- Use interactive dynamic software to generate images
- Spend a lot of time developing a concept - a slow start can lead to a fast finish
- Provide different experiences with the same concept
- Have students verbalize how the concepts are connected

Resources

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- burrill@msu.edu

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